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THE
Mathematical Principles
OF
NATURAL PHILOSOPHY,

BY
SIR ISAAC NEWTON;

Translated into English

BY **ANDREW MOTTE:**

TO WHICH ARE ADDED,

NEWTON'S SYSTEM OF THE WORLD;

A Short Comment on, and Defence of,

The Principia,

BY **WILLIAM EMERSON;**

WITH THE

LAWS OF THE MOON'S MOTION

According to Gravity,

BY **JOHN MACHIN,**

Astronomical Professor at Gresham College, and Secretary to the Royal Society.

A new Edition,

(WITH THE LIFE OF THE AUTHOR; AND A PORTRAIT, TAKEN FROM THE BUST IN
THE ROYAL OBSERVATORY AT GREENWICH)

CAREFULLY REVISED AND CORRECTED BY

WILLIAM DAVIS,

Author of a "Treatise on Land Surveying," "Use of the Globes," "Keys to Bonnycastle's
Arithmetic, Algebra, and Mensuration," Editor of the "Mathematical Companion,"

"Rowe, Simpson, and Maclaurin's Fluxions," "Fenning's Algebraist's
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THE
MATHEMATICAL PRINCIPLES
OF
NATURAL PHILOSOPHY.

OF THE MOTION OF BODIES.

BOOK II.

SECTION I.

Of the motion of bodies that are resisted in the ratio of the velocity.

PROPOSITION I. THEOREM I.

If a body is resisted in the ratio of its velocity, the motion lost by resistance is as the space gone over in its motion.

FOR since the motion lost in each equal particle of time is as the velocity, that is, as the particle of space gone over, then, by composition, the motion lost in the whole time will be as the whole space gone over. Q.E.D.

COR. Therefore if the body, destitute of all gravity, move by its innate force only in free spaces, and there be given both its whole motion at the beginning, and also the motion remaining after some part of the way is gone over, there will be given also the whole space which the body can describe in an infinite time. For that space will be to the space now described as the whole motion at the beginning is to the part lost of that motion.

LEMMA I.

Quantities proportional to their differences are continually proportional.

Let A be to $A - B$ as B to $B - C$ and C to $C - D$, &c. and, by conversion, A will be to B as B to C and C to D, &c. Q.E.D.

PROPOSITION II. THEOREM II.

If a body is resisted in the ratio of its velocity, and moves, by its vis infinita only, through a similar medium, and the times be taken equal, the velocities in the beginning of each of the times are in a geometrical progression, and the spaces described in each of the times are as the velocities. (Pl. 1, Fig. 1.)

CASE 1. Let the time be divided into equal particles; and if at the very beginning of each particle we suppose the resistance to act with one single impulse which is as the velocity, the decrement of the velocity in each of the particles of time will be as the same velocity. Therefore the velocities are proportional to their differences, and therefore (by lem. 1, book 2) continually proportional. Therefore if out of an equal number of particles there be compounded any equal portions of time, the velocities at the beginning of those times will be as terms in a continued progression, which are taken by intervals, omitting every where an equal number of intermediate terms. But the ratios of these terms are compounded of the equal ratios of the intermediate terms equally repeated, and therefore are equal. Therefore the velocities, being proportional to those terms, are in geometrical progression. Let those equal particles of time be diminished, and their number increased *in infinitum*, so that the impulse of resistance may become continual; and the velocities at the beginnings of equal times, always continually proportional, will be also in this case continually proportional. Q.E.D.

CASE 2. And, by division, the differences of the velocities, that is, the parts of the velocities lost in each of the times, are as the wholes; but the spaces described in each of the times are as the lost parts of the velocities (by prop. 1, book 2), and therefore are also as the wholes. Q.E.D.

COROL. Hence if to the rectangular asymptotes AC, CH, the hyperbola BG is described, and AB, DG, be drawn perpendicular to the asymptote AC, and both the velocity of the body, and the resistance of the medium, at the very beginning of the motion, be expressed by any given line AC, and, after some time is elapsed, by the indefinite line DC; the time may be expressed by the area ABGD, and the space described in that time by the line AD. For if that area, by the motion of the point D, be uniformly increased in the same manner as the time, the right line DC will decrease in a geometrical ratio in the same manner as the velocity; and the parts of the right line AC, described in equal times, will decrease in the same ratio.

PROPOSITION III. PROBLEM I.

To define the motion of a body which, in a similar medium, ascends or descends in a right line, and is resisted in the ratio of its velocity, and acted upon by an uniform force of gravity.
(Pl. 1, Fig. 2.)

The body ascending, let the gravity be expounded by any given rectangle BACH; and the resistance of the medium, at the beginning of the ascent, by the rectangle BADE, taken on the contrary side of the right line AB. Through the point B, with the rectangular asymptotes AC, CH, describe an hyperbola, cutting the perpendiculars DE, de, in G, g; and the body ascending will in the time DGgd describe the space EGge; in the time DGBA, the space of the whole ascent EGB; in the time ABKI, the space of descent BFK; and in the time IKki the space of descent KFFk; and the velocities of the bodies (proportional to the resistance of the medium) in these periods of time will be ABED, ABed, 0, ABFI, ABfi, respectively; and the greatest velocity which the body can acquire by descending will be BACH.

For let the rectangle BACH (Pl. 1, Fig. 3) be resolved into innumerable rectangles Ak, Kl, Lm, Mn, &c. which shall be as the increments of the velocities produced in so many equal times; then will 0, Ak, Al, Am, An, &c. be as the whole velocities, and therefore (by supposition) as the resistances of the medium in the beginning of each of the equal

times. Make AC to AK, or ABHC to ABkK, as the force of gravity to the resistance in the beginning of the second time; then from the force of gravity subduct the resistances, and ABHC, KkHC, LlHC, MmHC, &c. will be as the absolute forces with which the body is acted upon in the beginning of each of the times, and therefore (by law 2) as the increments of the velocities, that is, as the rectangles Ak, Kl, Lm, Mn, &c. and therefore (by lem. 1, book 2) in a geometrical progression. Therefore, if the right lines Kk, Ll, Mm, Nn, &c. are produced so as to meet the hyperbola in q, r, s, t, &c. the areas ABqK, KqrL, LrsM, MstN, &c. will be equal, and therefore analogous to the equal times and equal gravitating forces. But the area ABqK (by corol. 3, lem. 7 and 8, book 1) is to the area Bkq as Kq to $\frac{1}{2}kq$, or AC to $\frac{1}{2}K$, that is, as the force of gravity to the resistance in the middle of the first time. And by the like reasoning the areas qKLr, rLMs, sMNt, &c. are to the areas qklr, rlms, smnt, &c. as the gravitating forces to the resistances in the middle of the second, third, fourth time, and so on. Therefore since the equal areas BAKq, qKLr, rLMs, sMNt, &c. are analogous to the gravitating forces, the areas Bkq, qklr, rlms, smnt, &c. will be analogous to the resistances in the middle of each of the times, that is (by supposition), to the velocities, and so to the spaces described. Take the sums of the analogous quantities, and the areas Bkq, Blr, Bms, Bnt, &c. will be analogous to the whole spaces described; and also the areas ABqK, ABrL, ABsM, ABtN, &c. to the times. Therefore the body, in descending, will in any time ABrL describe the space Blr, and in the time LrtN the space rlnt. Q.E.D. And the like demonstration holds in ascending motion.

COROL. 1. Therefore the greatest velocity that the body can acquire by falling is to the velocity acquired in any given time, as the given force of gravity which perpetually acts upon it to the resisting force which opposes it at the end of that time.

COROL. 2. But the time being augmented in an arithmetical progression, the sum of that greatest velocity and the

velocity in the ascent, and also their difference in the descent, decreases in a geometrical progression.

COROL. 3. Also the differences of the spaces, which are described in equal differences of the times, decrease in the same geometrical progression.

COROL. 4. The space described by the body is the difference of two spaces, whereof one is as the time taken from the beginning of the descent, and the other as the velocity; which [spaces] also at the beginning of the descent are equal among themselves.

PROPOSITION IV. PROBLEM II.

Supposing the force of gravity in any similar medium to be uniform, and to tend perpendicularly to the plane of the horizon; to define the motion of a projectile therein, which suffers resistance proportional to its velocity. (Pl. 1, Fig. 4.)

Let the projectile go from any place D in the direction of any right line DP, and let its velocity at the beginning of the motion be expounded by the length DP. From the point P let fall the perpendicular PC on the horizontal line DC, and cut DC in A, so that DA may be to AC as the resistance of the medium arising from the motion upwards at the beginning to the force of gravity; or (which comes to the same) so that the rectangle under DA and DP may be to that under AC and CP as the whole resistance at the beginning of the motion to the force of gravity. With the asymptotes DC, CP, describe any hyperbola GTBS cutting the perpendiculars DG, AB, in G and B; complete the parallelogram DGKC, and let its side GK cut AB in Q. Take a line N in the same ratio to QB as DC is in to CP; and from any point R of the right line DC erect RT perpendicular to it, meeting the hyperbola in T, and the right lines EH, GK, DP, in I, t, and V; in that perpendicular take Vr equal to $\frac{tGT}{N}$, or, which is

the same thing, take Rr equal to $\frac{GTIE}{N}$; and the projectile in the time DRTG will arrive at the point r, describing the curve line DraF, the locus of the point r; thence it will come to its greatest height a in the perpendicular AB; and after-

wards ever approach to the asymptote PC. And its velocity in any point r will be as the tangent rL to the curve. Q.E.I.

For N is to QB as DC to CP or DR to RV , and therefore RV is equal to $\frac{DR \times QB}{N}$, and Rr (that is, $RV - Vr$, or $\frac{DR \times QB - tGT}{N}$) is equal to $\frac{DR \times AB - RDGT}{N}$. Now,

let the time be expounded by the area $RDGT$, and (by laws, cor. 2) distinguish the motion of the body into two others, one of ascent, the other lateral. And since the resistance is as the motion, let that also be distinguished into two parts proportional and contrary to the parts of the motion: and therefore the length described by the lateral motion will be (by prop. 2, book 2) as the line DR , and the height (by prop. 3, book 2) as the area $DR \times AB - RDGT$, that is, as the line Rr . But in the very beginning of the motion the area $RDGT$ is equal to the rectangle $DR \times AQ$, and therefore that line Rr

(or $\frac{DR \times AB - DR \times AQ}{N}$) will then be to DR as $AB - AQ$ or QB to N , that is, as CP to DC ; and therefore as the motion upwards to the motion lengthwise at the beginning. Since, therefore, Rr is always as the height, and DR always as the length, and Rr is to DR at the beginning as the height to the length, it follows, that Rr is always to DR as the height to the length; and therefore that the body will move in the line $DraF$, which is the locus of the point r . Q.E.D.

COR. 1. Therefore Rr is equal to $\frac{DR \times AB}{N} - \frac{RDGT}{N}$; and therefore if RT be produced to X so that RX may be equal to $\frac{DR \times AB}{N}$, that is, if the parallelogram $ACPY$ be completed, and Dy cutting CP in Z be drawn, and RT be produced till it meets DY in X ; Xr will be equal to $\frac{RDGT}{N}$, and therefore proportional to the time.

COR. 2. Whence if innumerable lines CR , or, which is the same, innumerable lines ZX , be taken in a geometrical

progression, there will be as many lines Xr in an arithmetical progression. And hence the curve $DraF$ is easily delineated by the table of logarithms.

COR. 3. If a parabola be constructed to the vertex D (Pl. 1, Fig. 5), and the diameter DG produced downwards, and its latus rectum is to $2DP$ as the whole resistance at the beginning of the motion to the gravitating force, the velocity with which the body ought to go from the place D , in the direction of the right line DP , so as in an uniform resisting medium to describe the curve $DraF$, will be the same as that with which it ought to go from the same place D , in the direction of the same right line DP , so as to describe a parabola in a non-resisting medium. For the latus rectum of this parabola, at the very beginning of the motion, is $\frac{DV^2}{V_r}$; and V_r is $\frac{tGT}{N}$ or

$\frac{DR \times Tt}{2N}$. But a right line, which, if drawn, would touch the hyperbola GTS in G , is parallel to DK , and therefore Tt is $\frac{CK \times DR}{DC}$, and N is $\frac{QB \times DC}{CP}$. And therefore V_r is equal to $\frac{DR^2 \times CK \times CP}{2DC^2 \times QB}$, that is (because DR and DC ,

DV and DP , are proportionals), to $\frac{DV^2 \times CK \times CP}{2DP^2 \times QB}$; and the latus rectum $\frac{DV^2}{V_r}$ comes out $\frac{2DP^2 \times QB}{CK \times CP}$, that is (because

QB and CK , DA and AC , are proportional), $\frac{2DP^2 \times DA}{AC \times CP}$, and therefore is to $2DP$ as $DP \times DA$ to $CP \times AC$; that is, as the resistance to the gravity. Q.E.D.

COR. 4. Hence if a body be projected from any place D (Pl. 1, Fig. 4) with a given velocity, in the direction of a right line DP given by position, and the resistance of the medium, at the beginning of the motion, be given, the curve $DraF$, which that body will describe, may be found. For the velocity being given, the latus rectum of the parabola is given, as is well known. And taking $2DP$ to that latus rectum, as the force of gravity to the resisting force, DP is also given.

Then cutting DC in A , so that $CP \times AC$ may be to $DP \times DA$ in the same ratio of the gravity to the resistance, the point A will be given. And hence the curve $DraF$ is also given.

COR. 5. And, on the contrary, if the curve $DraF$ be given, there will be given both the velocity of the body and the resistance of the medium in each of the places r . For the ratio of $CP \times AC$ to $DP \times DA$ being given, there is given both the resistance of the medium at the beginning of the motion, and the latus rectum of the parabola; and thence the velocity at the beginning of the motion is given also. Then from the length of the tangent rL there is given both the velocity proportional to it, and the resistance proportional to the velocity in any place r .

COR. 6. But since the length $2DP$ is to the latus rectum of the parabola as the gravity to the resistance in D ; and, from the velocity augmented, the resistance is augmented in the same ratio, but the latus rectum of the parabola is augmented in the duplicate of that ratio, it is plain that the length $2DP$ is augmented in that simple ratio only, and is therefore always proportional to the velocity; nor will it be augmented or diminished by the change of the angle CDP , unless the velocity be also changed.

COR. 7. Hence appears the method of determining the curve $DraF$ (Pl. 1, Fig. 6) nearly, from the phenomena, and thence collecting the resistance and velocity with which the body is projected. Let two similar and equal bodies be projected with the same velocity, from the place D , in different angles CDP , CDp ; and let the places F , f , where they fall upon the horizontal plane DC , be known. Then, taking any length for DP or Dp , suppose the resistance in D to be to the gravity in any ratio whatsoever, and let that ratio be expounded by any length SM . Then, by computation, from that assumed length DP , find the lengths DF , Df ; and from the ratio $\frac{Ff}{DF}$, found by calculation, subduct the same ratio as found by experiment; and let the difference be expounded by the perpendicular MN . Repeat the same a second and a third time, by assuming always a new ratio SM of the resistance to the gravity, and collecting a new difference MN . Draw the

affirmative differences on one side of the right line SM, and the negative on the other side; and through the points N,N,N, draw a regular curve NNN, cutting the right line SMMM in X, and SX will be the true ratio of the resistance to the gravity, which was to be found. From this ratio the length DF is to be collected by calculation; and a length, which is to the assumed length DP as the length DF known by experiment to the length DF just now found, will be the true length DP. This being known, you will have both the curve line DraF which the body describes, and also the velocity and resistance of the body in each place.

SCHOLIUM.

But, yet, that the resistance of bodies is in the ratio of the velocity, is more a mathematical hypothesis than a physical one. In mediums void of all tenacity, the resistances made to bodies are in the duplicate ratio of the velocities. For, by the action of a swifter body, a greater motion in proportion to a greater velocity is communicated to the same quantity of the medium in a less time; and in an equal time, by reason of a greater quantity of the disturbed medium, a motion is communicated in the duplicate ratio greater; and the resistance (by law 2 and 3) is as the motion communicated. Let us, therefore, see what motions arise from this law of resistance.

SECTION II.

Of the motion of bodies that are resisted in the duplicate ratio of their velocities.

PROPOSITION V. THEOREM III.

If a body is resisted in the duplicate ratio of its velocity, and moves by its innate force only through a similar medium; and the times be taken in a geometrical progression, proceeding from less to greater terms: I say, that the velocities at the beginning of each of the times are in the same geometrical progression inversely; and that the spaces are equal which are described in each of the times. (Pl. 1, Fig. 7.)

For since the resistance of the medium is proportional to the square of the velocity, and the decrement of the velocity is proportional to the resistance; if the time be divided into in-

numerable equal particles, the squares of the velocities at the beginning of each of the times will be proportional to the differences of the same velocities. Let those particles of time be $AK, KL, LM, \&c.$ taken in the right line CD ; and erect the perpendiculars $AB, Kk, Ll, Mm, \&c.$ meeting the hyperbola $BklmG$, described with the centre C , and the rectangular asymptotes CD, CH , in $B, k, l, m, \&c.$; then AB will be to Kk as CK to CA , and, by division, $AB - Kk$ to Kk as AK to CA , and, alternately, $AB - Kk$ to AK as Kk to CA ; and therefore as $AB \times Kk$ to $AB \times CA$. Therefore since AK and $AB \times CA$ are given, $AB - Kk$ will be as $AB \times Kk$; and, lastly, when AB and Kk coincide, as AB^2 . And, by the like reasoning, $Kk - Ll, Ll - Mm, \&c.$ will be as $Kk^2, Ll^2, \&c.$ Therefore the squares of the lines $AB, Kk, Ll, Mm, \&c.$ are as their differences; and, therefore, since the squares of the velocities were shewn above to be as their differences, the progression of both will be alike. This being demonstrated, it follows also that the areas described by these lines are in a like progression with the spaces described by these velocities. Therefore if the velocity at the beginning of the first time AK be expounded by the line AB , and the velocity at the beginning of the second time KL by the line Kk , and the length described in the first time by the area $AKkB$, all the following velocities will be expounded by the following lines $Ll, Mm, \&c.$ and the lengths described by the areas $Kl, Lm, \&c.$ And, by composition, if the whole time be expounded by AM , the sum of its parts, the whole length described will be expounded by $AMmB$, the sum of its parts. Now conceive the time AM to be divided into the parts $AK, KL, LM, \&c.$ so that $CA, CK, CL, CM, \&c.$ may be in a geometrical progression; and these parts will be in the same progression, and the velocities $AB, Kk, Ll, Mm, \&c.$ will be in the same progression inversely, and the spaces described $Ak, Kl, Lm, \&c.$ will be equal. Q.E.D.

COR. 1. Hence it appears, that if the time be expounded by any part AD of the asymptote, and the velocity in the beginning of the time by the ordinate AB , the velocity at the end of the time will be expounded by the ordinate DG ; and the whole space described by the adjacent hyperbolic

area $ABGD$; and the space which any body can describe in the same time AD , with the first velocity AB , in a non-resisting medium, by the rectangle $AB \times AD$.

COR. 2. Hence the space described in a resisting medium is given, by taking it to the space described with the uniform velocity AB in a non-resisting medium, as the hyperbolic area $ABGD$ to the rectangle $AB \times AD$.

COR. 3. The resistance of the medium is also given, by making it equal, in the very beginning of the motion, to an uniform centripetal force, which could generate, in a body falling through a non-resisting medium, the velocity AB in the time AC . For if BT be drawn touching the hyperbola in B , and meeting the asymptote in T , the right line AT will be equal to AC , and will express the time in which the first resistance, uniformly continued, may take away the whole velocity AB .

COR. 4. And thence is also given the proportion of this resistance to the force of gravity, or any other given centripetal force.

COR. 5. And, *vice versa*, if there is given the proportion of the resistance to any given centripetal force, the time AC is also given, in which a centripetal force equal to the resistance may generate any velocity as AB ; and thence is given the point B , through which the hyperbola, having CH , CD , for its asymptotes, is to be described; as also the space $ABGD$, which a body, by beginning its motion with that velocity AB , can describe in any time AD , in a similar resisting medium.

PROPOSITION VI. THEOREM IV.

Homogeneous and equal spherical bodies, opposed by resistances that are in the duplicate ratio of the velocities, and moving on by their innate force only, will, in times which are reciprocally as the velocities at the beginning, describe equal spaces, and lose parts of their velocities proportional to the wholes. (Pl. 1, Fig. 8.)

To the rectangular asymptotes CD , CH , describe any hyperbola $BbEe$, cutting the perpendiculars AB , ab , DE , de , in B , b , E , e ; let the initial velocities be expounded by the perpendiculars AB , DE , and the times by the lines Aa , Dd .

Therefore as Aa is to Dd , so (by the hypothesis) is DE to AB , and so (from the nature of the hyperbola) is CA to CD ; and, by composition, so is Ca to Cd . Therefore the areas $ABba$, $DEed$, that is, the spaces described, are equal among themselves, and the first velocities AB , DE , are proportional to the last ab , de ; and therefore, by division, proportional to the parts of the velocities lost, $AB - ab$, $DE - de$. Q.E.D.

PROPOSITION VII. THEOREM V.

If spherical bodies are resisted in the duplicate ratio of their velocities, in times which are as the first motions directly, and the first resistances inversely, they will lose parts of their motions proportional to the wholes, and will describe spaces proportional to those times and the first velocities conjunctly.

For the parts of the motions lost are as the resistances and times conjunctly. Therefore, that those parts may be proportional to the wholes, the resistance and time conjunctly ought to be as the motion. Therefore the time will be as the motion directly and the resistance inversely. Wherefore the particles of the times being taken in that ratio, the bodies will always lose parts of their motions proportional to the wholes, and therefore will retain velocities always proportional to their first velocities. And because of the given ratio of the velocities, they will always describe spaces which are as the first velocities and the times conjunctly. Q.E.D.

COR. 1. Therefore if bodies equally swift are resisted in a duplicate ratio of their diameters, homogeneous globes moving with any velocities whatsoever, by describing spaces proportional to their diameters, will lose parts of their motions proportional to the wholes. For the motion of each globe will be as its velocity and mass conjunctly, that is, as the velocity and the cube of its diameter; the resistance (by supposition) will be as the square of the diameter and the square of the velocity conjunctly; and the time (by this proposition) is in the former ratio directly, and in the latter inversely, that is, as the diameter directly and the velocity inversely; and therefore the space, which is proportional to the time and velocity, is as the diameter.

COR. 2. If bodies equally swift are resisted in a sesquiplicate ratio of their diameters, homogeneous globes, moving with any velocities whatsoever, by describing spaces that are in a sesquiplicate ratio of the diameters, will lose parts of their motions proportional to the wholes.

COR. 3. And universally, if equally swift bodies are resisted in the ratio of any power of the diameters, the spaces, in which homogeneous globes moving with any velocity whatsoever, will lose parts of their motions proportional to the wholes, will be as the cubes of the diameters applied to that power. Let those diameters be D and E ; and if the resistances, where the velocities are supposed equal, are as D^n and E^n ; the spaces in which the globes, moving with any velocities whatsoever, will lose parts of their motions proportional to the wholes, will be as $D^3 - ^n$ and $E^3 - ^n$. And therefore homogeneous globes, in describing spaces proportional to $D^3 - ^n$ and $E^3 - ^n$, will retain their velocities in the same ratio to one another as at the beginning.

COR. 4. Now if the globes are not homogeneous, the space described by the denser globe must be augmented in the ratio of the density. For the motion, with an equal velocity, is greater in the ratio of the density, and the time (by this prop.) is augmented in the ratio of motion directly, and the space described in the ratio of the time.

COR. 5. And if the globes move in different mediums, the space, in a medium which, *cæteris paribus*, resists the most, must be diminished in the ratio of the greater resistance. For the time (by this prop.) will be diminished in the ratio of the augmented resistance, and the space in the ratio of the time.

LEMMA II.

The moment of any genitum is equal to the moments of each of the generating sides drawn into the indices of the powers of those sides, and into their coefficients continually.

I call any quantity a *genitum* which is not made by addition or subduction of divers parts, but is generated or produced in arithmetic by the multiplication, division, or extraction of the root of any terms whatsoever; in geometry by the

invention of contents and fides, or of the extremes and means of proportionals. Quantities of this kind are products, quotients, roots, rectangles, squares, cubes, square and cubic fides, and the like. These quantities I here consider as variable and indeterminated, and increasing or decreasing, as it were, by a perpetual motion or flux; and I understand their momentaneous increments or decrements by the name of moments; so that the increments may be esteemed as added or affirmative moments, and the decrements as subducted or negative ones. But take care not to look upon finite particles as such. Finite particles are not moments, but the very quantities generated by the moments. We are to conceive them as the just nascent principles of finite magnitudes. Nor do we in this lemma regard the magnitude of the moments, but their first proportion, as nascent. It will be the same thing, if, instead of moments, we use either the velocities of the increments and decrements (which may also be called the motions, mutations, and fluxions of quantities), or any finite quantities proportional to those velocities. The coefficient of any generating fide is the quantity which arises by applying the genitum to that fide.

Wherefore the sense of the lemma is, that if the moments of any quantities A, B, C , &c. increasing or decreasing by a perpetual flux, or the velocities of the mutations which are proportional to them, be called a, b, c , &c. the moment or mutation of the generated rectangle AB will be $aB + bA$; the moment of the generated content ABC will be $aBC + bAC + cAB$; and the moments of the generated powers $A^2, A^3, A^4, A^{\frac{1}{2}}, A^{\frac{3}{2}}, A^{\frac{1}{3}}, A^{\frac{2}{3}}, A^{-1}, A^{-2}, A^{-\frac{1}{2}}$, will be $2aA, 3aA^2, 4aA^3, \frac{1}{2}aA^{-\frac{1}{2}}, \frac{3}{2}aA^{\frac{1}{2}}, \frac{1}{3}aA^{-\frac{2}{3}}, \frac{2}{3}aA^{-\frac{1}{3}}, -aA^{-2}, -2aA^{-3}, -\frac{1}{2}aA^{-\frac{3}{2}}$ respectively; and, in general, that the moment of any power $A^{\frac{n}{m}}$ will be $\frac{n}{m}aA^{\frac{n-m}{m}}$. Also, that the moment of the generated quantity A^2B will be $2aAB + bA^2$; the moment of the generated quantity $A^3B^4C^2$ will be $3aA^2B^4C^2 + 4bA^3B^3C^2 + 2cA^3B^4C$; and the moment of

the generated quantity $\frac{A^3}{B^2}$, or A^3B^{-2} , will be $3aA^2B^{-2} - 2bA^3B^{-3}$; and so on. The lemma is thus demonstrated.

CASE 1. Any rectangle, as AB, augmented by a perpetual flux, when, as yet, there wanted of the fides A and B half their moments $\frac{1}{2}a$ and $\frac{1}{2}b$, was $A - \frac{1}{2}a$ into $B - \frac{1}{2}b$, or $AB - \frac{1}{2}aB - \frac{1}{2}bA + \frac{1}{4}ab$; but as soon as the fides A and B are augmented by the other half moments, the rectangle becomes $A + \frac{1}{2}a$ into $B + \frac{1}{2}b$, or $AB + \frac{1}{2}aB + \frac{1}{2}bA + \frac{1}{4}ab$. From this rectangle subduct the former rectangle, and there will remain the excess $aB + bA$. Therefore with the whole increments a and b of the fides, the increment $aB + bA$ of the rectangle is generated. Q.E.D.

CASE 2. Suppose AB always equal to G, and then the moment of the content ABC or GC (by case 1) will be $gC + cG$, that is (putting AB and $aB + bA$ for G and g), $aBC + bAC + cAB$. And the reasoning is the same for contents under ever so many fides. Q.E.D.

CASE 3. Suppose the fides A, B, and C, to be always equal among themselves; and the moment $aB + bA$ of A^2 , that is, of the rectangle AB, will be $2aA$; and the moment $aBC + bAC + cAB$ of A^3 , that is, of the content ABC, will be $3aA^2$. And by the same reasoning, the moment of any power A^n is naA^{n-1} . Q.E.D.

CASE 4. Therefore since $\frac{1}{A}$ into A is 1, the moment of $\frac{1}{A}$ drawn into A, together with $\frac{1}{A}$ drawn into a, will be the moment of 1, that is, nothing. Therefore the moment of $\frac{1}{A}$, or of A^{-1} , is $-\frac{a}{A^2}$. And generally, since $\frac{1}{A^n}$ into A^n is 1, the moment of $\frac{1}{A^n}$ drawn into A^n together with $\frac{1}{A^n}$ into naA^{n-1} will be nothing. And, therefore, the moment of $\frac{1}{A^n}$ or A^{-n} will be $-\frac{na}{A^{n+1}}$. Q.E.D.

CASE 5. And since $A^{\frac{1}{2}}$ into $A^{\frac{1}{2}}$ is A , the moment of $A^{\frac{1}{2}}$ drawn into $2A^{\frac{1}{2}}$ will be a (by case 3); and, therefore, the moment of $A^{\frac{1}{2}}$ will be $\frac{a}{2A^{\frac{1}{2}}}$ or $\frac{1}{2}aA^{-\frac{1}{2}}$. And, generally, putting A_n^m equal to B , then A^m will be equal to B^n , and therefore maA^{m-1} equal to nbB^{n-1} , and maA^{-1} equal to nbB^{-1} , or $nbA^{-\frac{m}{n}}$; and therefore $\frac{m}{n}aA^{\frac{m-n}{n}}$ is equal to b , that is, equal to the moment of A_n^m . Q.E.D.

CASE 6. Therefore the moment of any generated quantity A^mB^n is the moment of A^m drawn into B^n , together with the moment of B^n drawn into A^m , that is, $maA^{m-1}B^n + nbB^{n-1}A^m$; and that whether the indices m and n of the powers be whole numbers or fractions, affirmative or negative. And the reasoning is the same for contents under more powers. Q.E.D.

COR. 1. Hence in quantities continually proportional, if one term is given, the moments of the rest of the terms will be as the same terms multiplied by the number of intervals between them and the given term. Let A, B, C, D, E, F , be continually proportional; then if the term C is given, the moments of the rest of the terms will be among themselves as $-2A, -B, D, 2E, 3F$.

COR. 2. And if in four proportionals the two means are given, the moments of the extremes will be as those extremes. The same is to be understood of the sides of any given rectangle.

COR. 3. And if the sum or difference of two squares is given, the moments of the sides will be reciprocally as the sides.

SCHOLIUM.

In a letter of mine to Mr. *J. Collins*, dated *December 10, 1672*, having described a method of tangents, which I suspected to be the same with *Slusius's* method, which at that time was not made public, I subjoined these words: *This is one particular, or rather a corollary, of a general method, which extends itself, without any troublesome calculation, not*

only to the drawing of tangents to any curve lines, whether geometrical or mechanical, or any how respecting right lines or other curves, but also to the resolving other abstruser kinds of problems about the crookedness, areas, lengths, centres of gravity of curves, &c. ; nor is it (as Hudden's method de Maximis & Minimis) limited to equations which are free from surd quantities. This method I have interwoven with that other of working in equations, by reducing them to infinite series. So far that letter. And these last words relate to a treatise I composed on that subject in the year 1671. The foundation of that general method is contained in the preceding lemma.

PROPOSITION VIII. THEOREM VI.

If a body in an uniform medium, being uniformly acted upon by the force of gravity, ascends or descends in a right line ; and the whole space described be distinguished into equal parts, and in the beginning of each of the parts (by adding or subtracting the resisting force of the medium to or from the force of gravity, when the body ascends or descends) you collect the absolute forces ; I say, that those absolute forces are in a geometrical progression. (Pl. 2, Fig. 1.)

For let the force of gravity be expounded by the given line AC; the force of resistance by the indefinite line AK; the absolute force in the descent of the body by the difference KC; the velocity of the body by a line AP, which shall be a mean proportional between AK and AC, and therefore in a subduplicate ratio of the resistance; the increment of the resistance made in a given particle of time by the lineolæ KL, and the contemporaneous increment of the velocity by the lineolæ PQ; and with the centre C, and rectangular asymptotes CA, CH, describe any hyperbola BNS meeting the erected perpendiculars AB, KN, LO, in B, N, and O. Because AK is as AP^2 , the moment KL of the one will be as the moment $2APQ$ of the other, that is, as $AP \times KC$; for the increment PQ of the velocity is (by law 2) proportional to the generating force KC. Let the ratio of KL be compounded with the ratio of KN, and the rectangle $KL \times KN$ will become as $AP \times KC \times KN$; that is (because the rectangle $KC \times KN$ is given), as AP. But the ultimate ratio of

the hyperbolic area KNOL to the rectangle $KL \times KN$ becomes, when the points K and L coincide, the ratio of equality. Therefore that hyperbolic evanescent area is as AP. Therefore the whole hyperbolic area ABOL is composed of particles KNOL which are always proportional to the velocity AP; and therefore is itself proportional to the space described with that velocity. Let that area be now divided into equal parts, as ABMI, IMNK, KNOL, &c., and the absolute forces AC, IC, KC, LC, &c. will be in a geometrical progression. Q.E.D. And by a like reasoning, in the ascent of the body, taking, on the contrary side of the point A, the equal areas ABmi, imnk, knol, &c. it will appear that the absolute forces AC, iC, kC, lC, &c. are continually proportional. Therefore if all the spaces in the ascent and descent are taken equal, all the absolute forces IC, kC, iC, AC, IC, KC, LC, &c. will be continually proportional. Q.E.D.

COR. 1. Hence if the space described be expounded by the hyperbolic area ABNK, the force of gravity, the velocity of the body, and the resistance of the medium, may be expounded by the lines AC, AP, and AK respectively; and *vice versa*.

COR. 2. And the greatest velocity which the body can ever acquire in an indefinite descent will be expounded by the line AC.

COR. 3. Therefore if the resistance of the medium answering to any given velocity be known, the greatest velocity will be found, by taking it to that given velocity in a ratio subduplicate of the ratio which the force of gravity bears to that known resistance of the medium.

PROPOSITION IX. THEOREM VII.

Supposing what is above demonstrated, I say, that if the tangents of the angles of the sector of a circle, and of an hyperbola, be taken proportional to the velocities, the radius being of a fit magnitude, all the time of the ascent to the highest place will be as the sector of the circle, and all the time of descending from the highest place as the sector of the hyperbola. (Pl. 2, Fig. 2.)

To the right line AC, which expresses the force of gravity, let AD be drawn perpendicular and equal. From the centre D with the semidiameter AD describe as well the quadrant

AtE of a circle, as the rectangular hyperbola AVZ, whose axis is AK, principal vertex A, and asymptote DC. Let Dp, DP be drawn; and the circular sector AtD will be as all the time of the ascent to the highest place; and the hyperbolic sector ATD as all the time of descent from the highest place; if so be that the tangents Ap, AP, of those sectors be as the velocities. (Pl. 2, Fig. 2.)

CASE 1. Draw Dvq cutting off the moments or least particles tDv and qDp, described in the same time, of the sector ADt and of the triangle ADp. Since those particles (because of the common angle D) are in a duplicate ratio of the sines, the particle tDv will be as $\frac{qDp \times tD^2}{pD^2}$, that is (because tD is given), as $\frac{qDp}{pD^2}$. But pD^2 is $AD^2 + Ap^2$, that is, $AD^2 + AD \times Ak$, or $AD \times Ck$; and qDp is $\frac{1}{2}AD \times pq$. Therefore tDv, the particle of the sector, is as $\frac{pq}{Ck}$; that is, as the least decrement pq of the velocity directly, and the force Ck, which diminishes the velocity, inversely; and therefore as the particle of time answering to the decrement of the velocity. And, by composition, the sum of all the particles tDv in the sector ADt will be as the sum of the particles of time answering to each of the lost particles pq of the decreasing velocity Ap, till that velocity, being diminished into nothing, vanishes; that is, the whole sector ADt is as the whole time of ascent to the highest place. Q.E.D.

CASE 2. Draw DQV cutting off the least particles TDV and PDQ of the sector DAV, and of the triangle DAQ; and these particles will be to each other as DT^2 to DP^2 , that is (if TX and AP are parallel), as DX^2 to DA^2 or TX^2 to AP^2 ; and, by division, as $DX^2 - TX^2$ to $DA^2 - AP^2$. But, from the nature of the hyperbola, $DX^2 - TX^2$ is AD^2 ; and, by the supposition, AP^2 is $AD \times AK$. Therefore the particles are to each other as AD^2 to $AD^2 - AD \times AK$; that is, as AD to $AD - AK$ or AC to CK: and therefore the particle TDV of the sector is $\frac{PDQ \times AC}{CK}$; and therefore (because AC

and AD are given) as $\frac{PQ}{CK}$; that is, as the increment of the velocity directly, and as the force generating the increment inversely; and therefore as the particle of the time answering to the increment. And, by composition, the sum of the particles of time, in which all the particles PQ of the velocity AP are generated, will be as the sum of the particles of the sector ATD; that is, the whole time will be as the whole sector. Q.E.D.

COR. 1. Hence if AB be equal to a fourth part of AC, the space which a body will describe by falling in any time will be to the space which the body could describe, by moving uniformly on in the same time with its greatest velocity AC, as the area ABNK, which expresses the space described in falling, to the area ATD, which expresses the time. For since AC is to AP as AP to AK, then (by cor. 1, lem. 2, of this book) LK is to PQ as 2AK to AP, that is, as 2AP to AC, and thence LK is to $\frac{1}{2}PQ$ as AP to $\frac{1}{2}AC$ or AB; and KN is to AC or AD as AB to CK; and, therefore, *ex æquo*, LKNO to DPQ as AP to CK. But DPQ was to DTV as CK to AC. Therefore, *ex æquo*, LKNO is to DTV as AP to AC; that is, as the velocity of the falling body to the greatest velocity which the body by falling can acquire. Since, therefore, the moments LKNO and DTV of the areas ABNK and ATD are as the velocities, all the parts of those areas generated in the same time will be as the spaces described in the same time; and therefore the whole areas ABNK and ADT, generated from the beginning, will be as the whole spaces described from the beginning of the descent. Q.E.D.

COR. 2. The same is true also of the space described in the ascent. That is to say, that all that space is to the space described in the same time, with the uniform velocity AC, as the area ABnk is to the sector ADt.

COR. 3. The velocity of the body, falling in the time ATD, is to the velocity which it would acquire in the same time in a non-resisting space, as the triangle APD to the hyperbolic sector ATD. For the velocity in a non-resisting medium would be as the time ATD, and in a resisting medium is as

AP, that is, as the triangle APD. And those velocities, at the beginning of the descent, are equal among themselves, as well as those areas ATD, APD.

COR. 4. By the same argument, the velocity in the ascent is to the velocity with which the body in the same time, in a non-resisting space, would lose all its motion of ascent, as the triangle ApD to the circular sector AtD; or as the right line Ap to the arc At.

COR. 5. Therefore the time in which a body, by falling in a resisting medium, would acquire the velocity AP, is to the time in which it would acquire its greatest velocity AC, by falling in a non-resisting space, as the sector ADT to the triangle ADC: and the time in which it would lose its velocity Ap, by ascending in a resisting medium, is to the time in which it would lose the same velocity by ascending in a non-resisting space, as the arc At to its tangent Ap.

COR. 6. Hence from the given time there is given the space described in the ascent or descent. For the greatest velocity of a body descending *in infinitum* is given (by corol. 2 and 3, theor. 6, of this book); and thence the time is given in which a body would acquire that velocity by falling in a non-resisting space. And taking the sector ADT or ADt to the triangle ADC in the ratio of the given time to the time just now found, there will be given both the velocity AP or Ap, and the area ABNK or ABnk, which is to the sector ADT, or ADt, as the space sought to the space which would, in the given time, be uniformly described with that greatest velocity found just before.

COR. 7. And by going backward, from the given space of ascent or descent ABnk or ABNK, there will be given the time ADt or ADT.

PROPOSITION X. PROBLEM III.

Suppose the uniform force of gravity to tend directly to the plane of the horizon, and the resistance to be as the density of the medium and the square of the velocity conjunctly; it is proposed to find the density of the medium in each place, which shall make the body move in any given curve line; the velocity of the body and the resistance of the medium in each place. (Pl. 2, Fig. 3.)

Let PQ be a plane perpendicular to the plane of the scheme itself; $PFHQ$ a curve line meeting that plane in the points P and Q ; G, H, I, K , four places of the body going on in this curve from F to Q ; and GB, HC, ID, KE , four parallel ordinates let fall from these points to the horizon, and standing on the horizontal line PQ at the points B, C, D, E ; and let the distances BC, CD, DE , of the ordinates be equal among themselves. From the points G and H let the right lines GL, HN , be drawn touching the curve in G and H , and meeting the ordinates CH, DI , produced upwards, in L and N ; and complete the parallelogram $HCDM$. And the times in which the body describes the arcs GH, HI , will be in a subduplicate ratio of the altitudes LH, NI , which the bodies would describe in those times, by falling from the tangents; and the velocities will be as the lengths described GH, HI directly, and the times inversely. Let the times be expounded by T and t , and the velocities by $\frac{GH}{T}$ and $\frac{HI}{t}$; and the decrement of the velocity produced in the time t will be expounded by $\frac{GH}{T} - \frac{HI}{t}$. This decrement arises from the resistance which retards the body, and from the gravity which accelerates it. Gravity, in a falling body, which in its fall describes the space NI , produces a velocity with which it would be able to describe twice that space in the same time, as *Galileo* has demonstrated; that is, the velocity $\frac{2NI}{t}$; but if the body describes the arc HI , it augments that arc only by the length $HI - HN$ or $\frac{MI \times NI}{HI}$; and therefore generates only the velocity $\frac{2MI \times NI}{t \times HI}$. Let this velocity be added to the before-mentioned decrement, and we shall have the decrement of the velocity arising from the resistance alone, that is, $\frac{GH}{T} - \frac{HI}{t} + \frac{2MI \times NI}{t \times HI}$. Therefore since, in the same time, the action of gravity generates, in a falling body,

the velocity $\frac{2NI}{t}$, the resistance will be to the gravity as $\frac{GH}{T} - \frac{HI}{t} + \frac{2MI \times NI}{t \times HI}$ to $\frac{2NI}{t}$ or as $\frac{t \times GH}{T} - HI + \frac{2MI \times NI}{HI}$ to $2NI$.

Now for the abscissas CB, CD, CE, put $-o, o, 2o$. For the ordinate CH put P; and for MI put any series $Qo + Ro^2 + So^3$, &c. And all the terms of the series after the first, that is, $Ro^2 + So^3 +$, &c. will be NI; and the ordinates DI, EK, and BG, will be $P - Qo - Ro^2 - So^3 -$, &c. $P - 2Qo - 4Ro^2 - 8So^3 -$, &c. and $P + Qo - Ro^2 + So^3 -$, &c. respectively. And by squaring the differences of the ordinates BG — CH and CH — DI, and to the squares thence produced adding the squares of BC and CD themselves, you will have $oo + QQoo - 2QRo^3 +$, &c. and $oo + QQoo + 2QRo^3 +$, &c. the squares of the arcs GH, HI; whose roots

$$o \sqrt{1 + QQ} - \frac{QRo^3}{\sqrt{1 + QQ}}, \text{ and } o \sqrt{1 + QQ} + \frac{QRo^3}{\sqrt{1 + QQ}}$$

are the arcs GH and HI. Moreover, if from the ordinate CH there be subducted half the sum of the ordinates BG and DI, and from the ordinate DI there be subducted half the sum of the ordinates CH and EK, there will remain Roo and $Roo + 3So^3$, the versed sines of the arcs GI and HK. And these are proportional to the lineolæ LH and NI, and therefore in the duplicate ratio of the infinitely small times T and t:

and thence the ratio $\frac{t}{T}$ is $\sqrt{\frac{R + 3So}{R}}$ or $\frac{R + \frac{3}{2}So}{R}$; and

$\frac{t \times GH}{T} - HI + \frac{2MI \times NI}{HI}$, by substituting the values of

$\frac{t}{T}$, GH, HI, MI and NI just found, becomes $\frac{3Soo}{2R} \sqrt{1 + QQ}$.

And since $2NI$ is $2Roo$, the resistance will be now to the gravity as $\frac{3Soo}{2R} \sqrt{1 + QQ}$ to $2Roo$, that is, as $3S \sqrt{1 + QQ}$ to $4RR$.

And the velocity will be such, that a body going off there-
with from any place H, in the direction of the tangent HN,
would describe, in vacuo, a parabola, whose diameter is HC,
and its latus rectum $\frac{HN^2}{NI}$ or $\frac{1 + QQ}{R}$.

And the resistance is as the density of the medium and the
square of the velocity conjunctly; and therefore the density of
the medium is as the resistance directly, and the square of
the velocity inversely; that is, as $\frac{3S \sqrt{1 + QQ}}{4RR}$ directly,
and $\frac{1 + QQ}{R}$ inversely; that is, as $\frac{S}{R \sqrt{1 + QQ}}$. Q.E.I.

COR. 1. If the tangent HN be produced both ways, so as
to meet any ordinate AF in T, $\frac{HT}{AC}$ will be equal to $\sqrt{1 + QQ}$,
and therefore in what has gone before may be put for $\sqrt{1 + QQ}$.
By this means the resistance will be to the gravity as $3S \times HT$
to $4RR \times AC$; the velocity will be as $\frac{HT}{AC \sqrt{R}}$, and the den-
sity of the medium will be as $\frac{S \times AC}{R \times HT}$.

COR. 2. And hence, if the curve line PFHQ be defined
by the relation between the base or abscissa AC and the or-
dinate CH, as is usual, and the value of the ordinate be re-
solved into a converging series, the problem will be expediti-
ously solved by the first terms of the series; as in the following
examples.

EXAMPLE 1. Let the line PFHQ be a semicircle describ-
ed upon the diameter PQ, to find the density of the medium
that shall make a projectile move in that line.

Bisect the diameter PQ in A; and call AQ, n; AC, a;
CH, e; and CD, o: then DI^2 or $AQ^2 - AD^2 = nn -$
 $aa - 2ao - oo$, or $ee - 2ao - oo$; and the root being ex-
tracted by our method, will give $DI = e - \frac{ao}{e} - \frac{oo}{2e} - \frac{aao}{2e^3}$
 $- \frac{ao^3}{2e^3} - \frac{a^3o^3}{2e^5} - \&c.$ Here put nn for $ee + aa$, and

DI will become $= e - \frac{ao}{e} - \frac{nnoo}{2e^3} - \frac{anno^3}{2e^5} - \dots$, &c.

Such series I distinguish into successive terms after this manner: I call that the first term in which the infinitely small quantity o is not found; the second, in which that quantity is of one dimension only; the third, in which it arises to two dimensions; the fourth, in which it is of three; and so *ad infinitum*. And the first term, which here is e , will always denote the length of the ordinate CH, standing at the beginning of the indefinite quantity o . The second term, which here is $\frac{ao}{e}$, will denote the difference between CH and DN; that is, the lineolæ MN which is cut off by completing the parallelogram HCDM, and therefore always determines the position of the tangent HN; as, in this case, by taking MN to HM as $\frac{ao}{e}$ to o , or a to e . The third term, which here is $\frac{nnoo}{2e^3}$, will represent the lineolæ IN, which lies between the tangent and the curve; and therefore determines the angles of contact IHN, or the curvature which the curve line has in H. If that lineolæ IN is of a finite magnitude, it will be expressed by the third term, together with those that follow *in infinitum*. But if that lineolæ be diminished *in infinitum*, the terms following become infinitely less than the third term, and therefore may be neglected. The fourth term determines the variation of the curvature; the fifth, the variation of the variation; and so on. Whence, by the way, appears no contemptible use of these series in the solution of problems that depend upon tangents, and the curvature of curves.

Now compare the series $e - \frac{ao}{e} - \frac{nnoo}{2e^3} - \frac{anno^3}{2e^5} - \dots$ with the series $P - Qo - Roo - So^3 - \dots$ and for P, Q, R , and S , put $e, \frac{a}{e}, \frac{nn}{2e^3}$ and $\frac{ann}{2e^3}$, and for $\sqrt{1 + QQ}$ put $\sqrt{1 + \frac{aa}{ee}}$ or $\frac{n}{e}$; and the density of the medium will come out as $\frac{a}{ne}$;

that is (because n is given), as $\frac{a}{e}$, or $\frac{AC}{CH}$, that is, as that length of the tangent HT , which is terminated at the semidiameter AF standing perpendicularly on PQ : and the resistance will be to the gravity as $3a$ to $2n$, that is, as $3AC$ to the diameter PQ of the circle; and the velocity will be as $\sqrt{CH}$. Therefore if the body goes from the place F , with a due velocity, in the direction of a line parallel to PQ , and the density of the medium in each of the places H is as the length of the tangent HT , and the resistance also in any place H is to the force of gravity as $3AC$ to PQ , that body will describe the quadrant FHQ of a circle. Q.E.I.

But if the same body should go from the place P , in the direction of a line perpendicular to PQ , and should begin to move in an arc of the semicircle PFQ , we must take AC or a on the contrary side of the centre A ; and therefore its sign must be changed, and we must put $-a$ for $+a$. Then the density of the medium would come out as $-\frac{a}{e}$. But nature does not admit of a negative density, that is, a density which accelerates the motion of bodies; and therefore it cannot naturally come to pass that a body by ascending from P should describe the quadrant PF of a circle. To produce such an effect, a body ought to be accelerated by an impelling medium, and not impeded by a resisting one.

EXAMPLE 2. Let the line PFQ be a parabola, having its axis AF perpendicular to the horizon PQ ; to find the density of the medium, which will make a projectile move in that line. (Pl. 2, Fig. 4).

From the nature of the parabola, the rectangle PDQ is equal to the rectangle under the ordinate DI and some given right line; that is, if that right line be called b ; PC , a ; PQ , c ; CH , e ; and CD , o ; the rectangle $a + o$ into $c - a - o$ or $ac - aa - 2ao + co - oo$, is equal to the rectangle b into DI , and therefore DI is equal to $\frac{ac - aa}{b} + \frac{c - 2a}{b}o - \frac{oo}{b}$.
Now the second term $\frac{c - 2a}{b}o$ of this series is to be put for

Q_0 , and the third term $\frac{00}{b}$ for R_{00} . But since there are no more terms, the coefficient S of the fourth term will vanish; and therefore the quantity $\frac{S}{R \sqrt{1 + QQ}}$, to which the density of the medium is proportional, will be nothing. Therefore, where the medium is of no density, the projectile will move in a parabola; as *Galileo* hath heretofore demonstrated. Q.E.I.

EXAMPLE 3. Let the line AGK be an hyperbola, having its asymptote NX perpendicular to the horizontal plane AK ; to find the density of the medium that will make a projectile move in that line. (Pl. 2, Fig. 5.)

Let MX be the other asymptote, meeting the ordinate DG produced in V ; and, from the nature of the hyperbola, the rectangle of XV into VG will be given. There is also given the ratio of DN to VX , and therefore the rectangle of DN into VG is given. Let that be bb : and, completing the parallelogram $DNXZ$, let BN be called a ; BD , o ; NX , c ; and let the given ratio of VZ to ZX or DN be $\frac{m}{n}$. Then DN will

be equal to $a - o$, VG equal to $\frac{bb}{a - o}$, VZ equal to $\frac{m}{n} \times a - o$,

and GD or $NX - VZ - VG$ equal to $c - \frac{m}{n} a + \frac{m}{n} o -$

$\frac{bb}{a - o}$. Let the term $\frac{bb}{a - o}$ be resolved into the converging

series $\frac{bb}{a} + \frac{bb}{aa} o + \frac{bb}{a^3} oo + \frac{bb}{a^4} o^3$, &c. and GD will be-

come equal to $c - \frac{m}{n} a - \frac{bb}{a} + \frac{m}{n} o - \frac{bb}{aa} o - \frac{bb}{a^3} o^2 -$

$\frac{bb}{a^4} o^3$, &c. The second term $\frac{m}{n} o - \frac{bb}{aa} o$ of this series is to

be used for Q_0 ; the third $\frac{bb}{a^3} o^2$, with its sign changed for

R_0^2 ; and the fourth $\frac{bb}{a^4} o^3$, with its sign changed also for S_0^2 ,

and their coefficients $\frac{m}{n} - \frac{bb}{aa}$, $\frac{bb}{a^3}$ and $\frac{bb}{a^4}$ are to be put for Q, R, and S in the former rule. Which being done, the density

of the medium will come out as $\frac{\frac{bb}{a^3} \sqrt{1 + \frac{mm}{nn} - \frac{2mbb}{naa} + \frac{b^4}{a^4}}}{\frac{bb}{a^4}}$

or $\frac{1}{\sqrt{aa + \frac{mm}{nn} aa - \frac{2mbb}{n} + \frac{b^4}{aa}}}$, that is, if in VZ you take

VY equal to VG, as $\frac{1}{XY}$. For aa and $\frac{m^2}{n^2} a^2 - \frac{2mbb}{n} + \frac{b^4}{aa}$ are the squares of XZ and ZY. But the ratio of the resistance to gravity is found to be that of 3XY to 2YG; and the velocity is that with which the body would describe a parabola, whose vertex is G, diameter DG, latus rectum $\frac{XY^2}{VG}$. Suppose,

therefore, that the densities of the medium in each of the places G are reciprocally as the distances XY, and that the resistance in any place G is to the gravity as 3XY to 2YG; and a body let go from the place A, with a due velocity, will describe that hyperbola AGK. Q.E.I.

EXAMPLE 4. Suppose, indefinitely, the line AGK to be an hyperbola described with the centre X, and the asymptotes MX, NX, so that, having constructed the rectangle XZDN, whose side ZD cuts the hyperbola in G and its asymptote in V, VG may be reciprocally as any power DN^n of the line ZX or DN, whose index is the number n: to find the density of the medium in which a projected body will describe this curve. (Pl. 2, Fig. 5.)

For BN, BD, NX, put A, O, C respectively, and let VZ be to XZ or DN as d to e, and VG be equal to $\frac{bb}{DN^n}$; then DN will be equal to A — O, $VG = \frac{bb}{(A-O)^n}$, $VZ = \frac{d}{e} A - O$, and GD or NX — VZ — VG equal to $C - \frac{d}{e} A + \frac{d}{e} O -$

$\frac{bb}{A - O^n}$. Let the term $\frac{bb}{A - O^n}$ be resolved into an infinite

series $\frac{bb}{A^n} + \frac{nbb}{A^{n+1}} \times O + \frac{nn+n}{2A^{n+2}} \times bbO^2 + \frac{n^3+3nn+2n}{6A^{n+3}}$

$\times bbO^3$, &c. and GD will be equal to $C - \frac{d}{e} A - \frac{bb}{A^n} + \frac{d}{e} O$

$- \frac{nbb}{A^{n+1}} O - \frac{+nn+n}{2A^{n+2}} bbO^2 - \frac{+n^3+3nn+2n}{6A^{n+3}} bbO^3$,

&c. The second term $\frac{d}{e} O - \frac{nbb}{A^{n+1}} O$ of this series is to be

used for Qo, the third $\frac{nn+n}{2A^{n+2}} bbO^2$ for Roo, the fourth

$\frac{n^3+3nn+2n}{6A^{n+3}} bbO^3$ for So³. And thence the density

of the medium $\frac{S}{R\sqrt{1+QQ}}$, in any place G, will be

$\frac{n+2}{3\sqrt{A^2 + \frac{dd}{ee} A^2 - \frac{2dnbb}{eA^n} A + \frac{nnb^4}{A^{2n}}}}$, and therefore if in VZ

you take VY equal to $n \times VG$, that density is reciprocally as

XY. For A^2 and $\frac{dd}{ee} A^2 - \frac{2dnbb}{eA^n} A + \frac{nnb^4}{A^{2n}}$ are the squares

of XZ and ZY. But the resistance in the same place G is to

the force of gravity as $3S \times \frac{XY}{A}$ to $4RR$, that is, as XY to

$\frac{2nn+2n}{n+2} VG$. And the velocity there is the same where-

with the projected body would move in a parabola, whose ver-

tex is G, diameter GD, and latus rectum $\frac{1+QQ}{R}$ or $\frac{2XY^2}{nn+n \times VG}$.

Q.E.I. *Q.E.I. Scholium*

SCHOLIUM.

In the same manner that the density of the medium comes

out to be as $\frac{S \times AC}{R \times HT}$, in cor. 1, if the resistance is put as

any power V^n of the velocity V, the density of the medium

will come out to be as $\frac{S}{R^{\frac{4-n}{2}}} \times \left[\frac{AC}{HT} \right]^{n-1}$. (Fig. 3.)

And therefore if a curve can be found, such that the ratio of $\frac{S}{R^{\frac{4-n}{2}}}$ to $\left[\frac{HT}{AC} \right]^{n-1}$, or of $\frac{S^2}{R^{4-n}}$ to $\left[1 + \frac{QQ}{1} \right]^{n-1}$, may be given; the body, in an uniform medium, whose resistance is as the power V^n of the velocity V , will move in this curve. But let us return to more simple curves.

Because there can be no motion in a parabola except in a non-resisting medium, but in the hyperbolas here described it is produced by a perpetual resistance; it is evident that the line which a projectile describes in an uniformly resisting medium approaches nearer to these hyperbolas than to a parabola. That line is certainly of the hyperbolic kind, but about the vertex it is more distant from the asymptotes, and in the parts remote from the vertex draws nearer to them than these hyperbolas here described. The difference, however, is not so great between the one and the other, but that these latter may be commodiously enough used in practice instead of the former. And perhaps these may prove more useful than an hyperbola that is more accurate, and at the same time more compounded. They may be made use of, then, in this manner. (Pl. 2, Fig. 5.)

Complete the parallelogram $XYGT$, and the right line GT will touch the hyperbola in G , and therefore the density of the medium in G is reciprocally as the tangent GT , and the velocity there as $\sqrt{\frac{GT^2}{GV}}$; and the resistance is to the force of gravity as GT to $\frac{2nn + 2n}{n + 2} + GV$.

Therefore if a body projected from the place A , in the direction of the right line AH (Pl. 2, Fig. 6), describes the hyperbola AGK , and AH produced meets the asymptote NX in H , and AI drawn parallel to it meets the other asymptote MX in I ; the density of the medium in A will be reciprocally as AH , and the velocity of the body as $\sqrt{\frac{AH^2}{AI}}$, and the resistance there

to the force of gravity as AH to $\frac{2nn + 2n}{n + 2} \times AI$. Hence the following rules are deduced.

RULE 1. If the density of the medium at A , and the velocity with which the body is projected, remain the same, and the angle NAH be changed; the lengths AH , AI , HX , will remain. Therefore if those lengths, in any one case, are found, the hyperbola may afterwards be easily determined from any given angle NAH .

RULE 2. If the angle NAH , and the density of the medium at A , remain the same, and the velocity with which the body is projected be changed, the length AH will continue the same; and AI will be changed in a duplicate ratio of the velocity reciprocally.

RULE 3. If the angle NAH , the velocity of the body at A , and the accelerative gravity, remain the same, and the proportion of the resistance at A to the motive gravity be augmented in any ratio; the proportion of AH to AI will be augmented in the same ratio, the latus rectum of the above mentioned parabola remaining the same, and also the length $\frac{AH^2}{AI}$ proportional to it; and therefore AH will be diminished in the same ratio, and AI will be diminished in the duplicate of that ratio. But the proportion of the resistance to the weight is augmented, when either the specific gravity is made less, the magnitude remaining equal, or when the density of the medium is made greater, or when, by diminishing the magnitude, the resistance becomes diminished in a less ratio than the weight.

RULE 4. Because the density of the medium is greater near the vertex of the hyperbola than it is in the place A , that a mean density may be preserved, the ratio of the least of the tangents GT to the tangent AH ought to be found, and the density in A augmented in a ratio a little greater than that of half the sum of those tangents to the least of the tangents GT .

RULE 5. If the lengths AH , AI , are given, and the figure AGK is to be described, produce HN to X , so that HX may

be to AI as $n + 1$ to 1; and with the centre X, and the asymptotes MX, NX, describe an hyperbola through the point A, such that AI may be to any of the lines VG as XV^n to XI^n .

RULE 6. By how much the greater the number n is, so much the more accurate are these hyperbolas in the ascent of the body from A, and less accurate in its descent to K; and the contrary. The conic hyperbola keeps a mean ratio between these, and is more simple than the rest. Therefore if the hyperbola be of this kind, and you are to find the point K, where the projected body falls upon any right line AN passing through the point A, let AN produced meet the asymptotes MX, NX, in M and N, and take NK equal to AM.

RULE 7. And hence appears an expeditious method of determining this hyperbola from the phænomena. Let two similar and equal bodies be projected with the same velocity, in different angles HAK, hAk (Pl. 2, Fig. 6), and let them fall upon the plane of the horizon in K and k; and note the proportion of AK to Ak. Let it be as d to e . Then erecting a perpendicular AI of any length, assume any how the length AH or Ah, and thence graphically, or by scale and compass, collect the lengths AK, Ak (by rule 6). If the ratio of AK to Ak be the same with that of d to e , the length of AH was rightly assumed. If not, take on the indefinite right line SM (Pl. 2, Fig. 7), the length SM equal to the assumed AH; and erect a perpendicular MN equal to the difference $\frac{AK}{Ak} - \frac{d}{e}$ of the ratios

drawn into any given right line. By the like method, from several assumed lengths AH, you may find several points N; and draw through them all a regular curve NNXN, cutting the right line SMMM in X. Lastly, assume AH equal to the abscissa SX, and thence find again the length AK; and the lengths, which are to the assumed length AI, and this last AH, as the length AK known by experiment to the length AK last found, will be the true lengths AI and AH, which were to be found. But these being given, there will be given also the resisting force of the medium in the place A, it being to

the force of gravity as AH to $2AI$. Let the density of the medium be increased by rule 4, and if the resisting force just found be increased in the same ratio, it will become still more accurate.

RULE 8. The lengths AH , HX being found; let there be now required the position of the line AH , according to which a projectile thrown with that given velocity shall fall upon any point K . At the points A and K (Pl. 2, Fig. 6) erect the lines AC , KF perpendicular to the horizon; whereof let AC be drawn downwards, and be equal to AI or $\frac{1}{2}HX$. With the asymptotes AK , KF , describe an hyperbola, whose conjugate shall pass through the point C ; and from the centre A , with the interval AH , describe a circle cutting that hyperbola in the point H ; then the projectile thrown in the direction of the right line AH will fall upon the point K . *Q.E.I.* For the point H , because of the given length AH , must be somewhere in the circumference of the described circle. Draw CH meeting AK and KF in E and F ; and because CH , MX are parallel, and AC , AI equal, AE will be equal to AM , and therefore also equal to KN . But CE is to AE as FH to KN , and therefore CE and FH are equal. Therefore the point H falls upon the hyperbolic curve described with the asymptotes AK , KF , whose conjugate passes through the point C ; and is, therefore, found in the common intersection of this hyperbolic curve and the circumference of the described circle. *Q.E.D.* It is to be observed that this operation is the same, whether the right line AKN be parallel to the horizon, or inclined thereto in any angle; and that from two intersections H , H , there arise two angles NAH , NAH ; and that, in mechanical practice, it is sufficient once to describe a circle, then to apply a ruler CH , of an indeterminate length, so to the point C , that its part FH , intercepted between the circle and the right line FK , may be equal to its part CE placed between the point C and the right line AK .

What has been said of hyperbolas may be easily applied to parabolas. For if (Pl. 2, Fig. 8) a parabola be represented by $XAGK$, touched by a right line XV in the vertex X , and the ordinates IA , VG be as any powers XI^n , XV^n , of the ab-

scissas XI, XV; draw XT, GT, AH, whereof let XT be parallel to VG, and let GT, AH touch the parabola in G and A; and a body projected from any place A, in the direction of the right line AH, with a due velocity, will describe this parabola, if the density of the medium in each of the places G be reciprocally as the tangent GT. In that case the velocity in G will be the same as would cause a body, moving in a non-resisting space, to describe a conic parabola, having G for its vertex, VG produced downwards for its diameter, and $\frac{2GT^2}{nn - n \times VG}$ for its latus rectum. And the resisting force

in G will be to the force of gravity as GT to $\frac{2nn - 2n}{n - 2} VG$.

Therefore, if NAK represent an horizontal line, and both the density of the medium at A, and the velocity with which the body is projected, remaining the same, the angle NAH be any how altered, the lengths AH, AI, HX will remain; and thence will be given the vertex X of the parabola, and the position of the right line XI; and by taking VG to IA as XV^a to XIⁿ, there will be given all the points G of the parabola, through which the projectile will pass.

SECTION III.

Of the motions of bodies which are resisted partly in the ratio of the velocities, and partly in the duplicate of the same ratio.

PROPOSITION XI. THEOREM VIII.

If a body be resisted partly in the ratio and partly in the duplicate ratio of its velocity, and moves in a similar medium by its innate force only; and the times be taken in arithmetical progression; then quantities reciprocally proportional to the velocities, increased by a certain given quantity, will be in geometrical progression. (Pl. 3, Fig. 1.)

With the centre C, and the rectangular asymptotes CADd and CH, describe an hyperbola BEe, and let AB, DE, de, be parallel to the asymptote CH. In the asymptote CD let A, G be given points; and, if the time be expounded by the hyperbolic area ABED uniformly increasing, I say, that the velocity may be expressed by the length DF, whose reciprocal

GD, together with the given line CG, compose the length CD, increasing in a geometrical progression.

For, let the areola DEed be the least given increment of the time, and Dd will be reciprocally as DE, and therefore directly as CD. Therefore the decrement of $\frac{1}{GD}$, which (by lem. 2, book 2) is $\frac{Dd}{GD^2}$, will be also as $\frac{CD}{GD^2}$ or $\frac{CG+GD}{GD^2}$, that

is, as $\frac{1}{GD} + \frac{CG}{GD^2}$. Therefore the time ABED uniformly increasing by the addition of the given particles EDde, it follows that $\frac{1}{GD}$ decreases in the same ratio with the velocity.

For the decrement of the velocity is as the resistance, that is (by the supposition), as the sum of two quantities, whereof one is as the velocity, and the other as the square of the velocity;

and the decrement of $\frac{1}{GD}$ is as the sum of the quantities $\frac{1}{GD}$ and $\frac{CG}{GD^2}$, whereof the first is $\frac{1}{GD}$ itself, and the last $\frac{CG}{GD^2}$.

is as $\frac{1}{GD^2}$; therefore $\frac{1}{GD}$ is as the velocity, the decrements of both being analogous. And if the quantity GD, reciprocally proportional to $\frac{1}{GD}$, be augmented by the given quantity CG; the sum CD, the time ABED uniformly increasing, will increase in a geometrical progression. Q.E.D.

COR. 1. Therefore, if, having the points A and G given, the time be expounded by the hyperbolic area ABED, the velocity may be expounded by $\frac{1}{GD}$, the reciprocal of GD.

COR. 2. And by taking GA to GD as the reciprocal of the velocity at the beginning to the reciprocal of the velocity at the end of any time ABED, the point G will be found. And that point being found, the velocity may be found from any other time given.

PROPOSITION XII. THEOREM IX.

The same things being supposed, I say, that if the spaces described are taken in arithmetical progression, the velocities augmented by a certain given quantity will be in geometrical progression. (Pl. 3, Fig. 2.)

In the asymptote CD let there be given the point R, and, erecting the perpendicular RS meeting the hyperbola in S, let the space described be expounded by the hyperbolic area RSED; and the velocity will be as the length GD, which, together with the given line CG, composes a length CD, decreasing in a geometrical progression, while the space RSED increases in an arithmetical progression.

For, because the increment EDde of the space is given, the lineolæ Dd, which is the decrement of GD, will be reciprocally as ED, and therefore directly as CD; that is, as the sum of the same GD and the given length CG. But the decrement of the velocity, in a time reciprocally proportional thereto, in which the given particle of space DdeE is described, is as the resistance and the time conjunctly, that is, directly as the sum of two quantities, whereof one is as the velocity, the other as the square of the velocity, and inversely as the velocity; and therefore directly as the sum of two quantities, one of which is given, the other is as the velocity. Therefore the decrement both of the velocity and the line GD is as a given quantity and a decreasing quantity conjunctly; and, because the decrements are analogous, the decreasing quantities will always be analogous; viz. the velocity, and the line GD. Q.E.D.

COR. 1. If the velocity be expounded by the length GD, the space described will be as the hyperbolic area DESR.

COR. 2. And, if the point R be assumed any how, the point G will be found, by taking GR to GD as the velocity at the beginning to the velocity after any space RSED is described. The point G being given, the space is given from the given velocity; and the contrary.

COR. 3. Whence, since (by prop. 11) the velocity is given from the given time, and (by this prop.) the space is given from the given velocity; the space will be given from the given time: and the contrary.

PROPOSITION XIII. THEOREM X.

Supposing that a body attracted downwards by an uniform gravity ascends or descends in a right line; and that the same is resisted partly in the ratio of its velocity, and partly in the duplicate ratio thereof: I say, that, if right lines parallel to the diameters of a circle and an hyperbola be drawn through the ends of the conjugate diameters, and the velocities be as some segments of those parallels drawn from a given point, the times will be as the sectors of the areas cut off by right lines drawn from the centre to the ends of the segments; and the contrary. (Pl. 3, Fig. 3.)

CASE 1. Suppose first that the body is ascending, and from the centre D, with any semi-diameter DB, describe a quadrant BETF of a circle, and through the end B of the semi-diameter DB draw the indefinite line BAP, parallel to the semi-diameter DF. In that line let there be given the point A, and take the segment AP proportional to the velocity. And since one part of the resistance is as the velocity, and another part as the square of the velocity, let the whole resistance be as $AP^2 + 2BAP$. Join DA, DP, cutting the circle in E and T, and let the gravity be expounded by DA^2 , so that the gravity shall be to the resistance in P as DA^2 to $AP^2 + 2BAP$; and the time of the whole ascent will be as the sector EDT of the circle.

For draw DVQ, cutting off the moment PQ of the velocity AP, and the moment DTV of the sector DET answering to a given moment of time; and that decrement PQ of the velocity will be as the sum of the forces of gravity DA^2 and of resistance $AP^2 + 2BAP$, that is (by 12 prop. 2 book, elem.), as DP^2 . Then the area DPQ, which is proportional to PQ, is as DP^2 , and the area DTV, which is to the area DPQ as DT^2 to DP^2 , is as the given quantity DT^2 . Therefore the area EDT decreases uniformly according to the rate of the future time, by subduction of given particles DTV, and is therefore proportional to the time of the whole ascent. Q.E.D.

CASE 2. If the velocity in the ascent of the body be expounded by the length AP as before, and the resistance be made as $AP^2 + 2BAP$, and if the force of gravity be less

than can be expressed by DA^2 ; take BD (Fig. 4) of such a length, that $AB^2 - BD^2$ may be proportional to the gravity, and let DF be perpendicular and equal to DB , and through the vertex F describe the hyperbola $FTVE$, whose conjugate semi-diameters are DB and DF , and which cuts DA in E , and DP , DQ in T and V ; and the time of the whole ascent will be as the hyperbolic sector TDE .

For the decrement PQ of the velocity, produced in a given particle of time, is as the sum of the resistance $AP^2 + 2BAP$ and of the gravity $AB^2 - BD^2$, that is, as $BP^2 - BD^2$. But the area DTV is to the area DPQ as DT^2 to DP^2 ; and, therefore, if GT be drawn perpendicular to DF , as GT^2 or $GD^2 - DF^2$ to BD^2 , and as GD^2 to BP^2 , and, by division, as DF^2 to $BP^2 - BD^2$. Therefore since the area DPQ is as PQ , that is, as $BP^2 - BD^2$, the area DTV will be as the given quantity DF^2 . Therefore the area EDT decreases uniformly in each of the equal particles of time, by the subduction of so many given particles DTV , and therefore is proportional to the time. Q.E.D.

CASE 3. Let AP be the velocity in the descent of the body, and $AP^2 + 2BAP$ the force of resistance, and $BD^2 - AB^2$ the force of gravity, the angle DBA being a right one. And if with the centre D , and the principal vertex B , there be described a rectangular hyperbola $BETV$ (Pl. 3, Fig. 5) cutting DA , DP , and DQ produced in E , T , and V ; the sector DET of this hyperbola will be as the whole time of descent.

For the increment PQ of the velocity, and the area DPQ proportional to it, is as the excess of the gravity above the resistance, that is, as $BD^2 - AB^2 - 2BAP - AP^2$ or $BD^2 - BP^2$. And the area DTV is to the area DPQ as DT^2 to DP^2 ; and therefore as GT^2 or $GD^2 - BD^2$ to BP^2 , and as GD^2 to BD^2 , and, by division, as BD^2 to $BD^2 - BP^2$. Therefore since the area DPQ is as $BD^2 - BP^2$, the area DTV will be as the given quantity BD^2 . Therefore the area EDT increases uniformly in the several equal particles of time by the addition of as many given particles DTV , and therefore is proportional to the time of the descent. Q.E.D.

COR. If with the centre D and the semi-diameter DA there be drawn through the vertex A an arc At similar to the arc ET, and similarly subtending the angle ADT, the velocity AP will be to the velocity which the body in the time EDT, in a non-resisting space, can lose in its ascent, or acquire in its descent, as the area of the triangle DAP to the area of the sector DAt; and therefore is given from the time given. For the velocity in a non-resisting medium is proportional to the time, and therefore to this sector; in a resisting medium, it is as the triangle; and in both mediums, where it is least, it approaches to the ratio of equality, as the sector and triangle do.

SCHOLIUM.

One may demonstrate also that case in the ascent of the body, where the force of gravity is less than can be expressed by DA^2 or $AB^2 + BD^2$, and greater than can be expressed by $AB^2 - DB^2$, and must be expressed by AB^2 . But I hasten to other things.

PROPOSITION XIV. THEOREM XI.

The same things being supposed, I say, that the space described in the ascent or descent is as the difference of the area by which the time is expressed, and of some other area which is augmented or diminished in an arithmetical progression; if the forces compounded of the resistance and the gravity be taken in a geometrical progression. (Pl. 3, Fig. 6, 7, 8.)

Take AC (in the three last figures) proportional to the gravity, and AK to the resistance; but take them on the same side of the point A, if the body is descending, otherwise on the contrary. Erect Ab, which make to DB as DB^2 to $4BAC$: and to the rectangular asymptotes CK, CH, describe the hyperbola bN; and, erecting KN perpendicular to CK, the area AbNK will be augmented or diminished in an arithmetical progression, while the forces CK are taken in a geometrical progression. I say, therefore, that the distance of the body from its greatest altitude is as the excess of the area AbNK above the area DET.

For, since AK is as the resistance, that is, as $AP^2 \times 2BAP$; assume any given quantity Z, and put AK equal to

$\frac{AP^2 + 2BAP}{Z}$; then (by lem. 2 of this book) the moment

KL of AK will be equal to $\frac{2APQ + 2BA \times PQ}{Z}$ or $\frac{2BPQ}{Z}$,

and the moment KLON of the area AbNK will be equal to

$\frac{2BPQ \times LO}{Z}$ or $\frac{BPQ \times BD^3}{2Z \times CK \times AB}$.

CASE 1. Now, if the body ascends, and the gravity be as $AB^2 + BD^2$, BET (in Fig. 6), being a circle, the line AC, which is proportional to the gravity, will be $\frac{AB^2 + BD^2}{Z}$, and

DP^2 or $AP^2 + 2BAP + AB^2 + BD^2$ will be $AK \times Z + AC \times Z$ or $CK \times Z$; and therefore the area DTV will be to the area DPQ as DT^2 or DB^2 to $CK \times Z$.

CASE 2. If the body ascends, and the gravity be as $AB^2 - BD^2$, the line AC (in Fig. 7) will be $\frac{AB^2 - BD^2}{Z}$, and DT^2 will be to DP^2 as DF^2 or DB^2 to $BP^2 - BD^2$ or $AP^2 + 2BAP + AB^2 - BD^2$, that is, to $AK \times Z + AC \times Z$ or $CK \times Z$. And therefore the area DTV will be to the area DPQ as DB^2 to $CK \times Z$.

CASE 3. And, by the same reasoning, if the body descends, and therefore the gravity is as $BD^2 - AB^2$, and the line AC (in Fig. 8) becomes equal to $\frac{BD^2 - AB^2}{Z}$, the area DTV will be to the area DPQ as DB^2 to $CK \times Z$: as above.

Since, therefore, these areas are always in this ratio, if for the area DTV, by which the moment of the time, always equal to itself, is expressed, there be put any determinate rect-angle, as $BD \times m$, the area DPQ, that is, $\frac{1}{2}BD \times PQ$, will be to $BD \times m$ as $CK \times Z$ to BD^2 . And thence $PQ \times BD^3$ becomes equal to $2BD \times m \times CK \times Z$, and the moment KLON of the area AbNK, found before, becomes $\frac{BP \times BD \times m}{AB}$. From the area DET subduct its moment

DTV or $BD \times m$, and there will remain $\frac{AP \times BD \times m}{AB}$. There-

fore the difference of the moments, that is, the moment of the difference of the areas, is equal to $\frac{AP \times BD \times m}{AB}$; and,

therefore (because of the given quantity $\frac{BD \times m}{AB}$), as the ve-

locity AP; that is, as the moment of the space which the body describes in its ascent or descent. And therefore the difference of the areas, and that space, increasing or decreasing by proportional moments, and beginning together or vanishing together, are proportional. Q.E.D.

COR. If the length, which arises by applying the area DET to the line BD, be called M; and another length V be taken in that ratio to the length M, which the line DA has to the line DE; the space which a body, in a resisting medium, describes in its whole ascent or descent, will be to the space which a body, in a non-resisting-medium, falling from rest, can describe in the same time, as the difference of the aforesaid areas to $\frac{BD \times V^2}{AB}$; and therefore is given from the time

given. For the space in a non-resisting medium is in a duplicate ratio of the time, or as V^2 ; and, because BD and AB are given,

as $\frac{BD \times V^2}{AB}$. This area is equal to the area $\frac{DA^2 \times BD \times M^2}{DE^2 \times AB}$

and the moment of M is m; and therefore the moment of

this area is $\frac{DA^2 \times BD \times 2M \times m}{DE^2 \times AB}$. But this moment is to the

moment of the difference of the aforesaid areas DET and

AbNK, viz. to $\frac{AP \times BD \times m}{AB}$, as $\frac{DA^2 \times BD \times M}{DE^2}$ to $\frac{1}{2}BD \times$

AP, or as $\frac{DA^2}{DE^2}$ into DET to DAP; and, therefore, when the

areas DET and DAP are least, in the ratio of equality. There-

fore the area $\frac{BD \times V^2}{AB}$ and the difference of the areas DET

and AbNK, when all these areas are least, have equal moments; and are therefore equal. Therefore since the velocities, and

therefore also the spaces in both mediums described together, in the beginning of the descent, or the end of the ascent, approach to equality, and therefore are then one to another as the area $\frac{BD \times V^2}{AB}$, and the difference of the areas DET and AbNK; and moreover since the space, in a non-resisting medium, is perpetually as $\frac{BD \times V^2}{AB}$, and the space, in a resisting medium, is perpetually as the difference of the areas DET and AbNK; it necessarily follows, that the spaces, in both mediums, described in any equal times, are one to another as that area $\frac{BD \times V^2}{AB}$, and the difference of the areas DET and AbNK. Q.E.D.

SCHOLIUM.

The resistance of spherical bodies in fluids arises partly from the tenacity, partly from the attrition, and partly from the density of the medium. And that part of the resistance which arises from the density of the fluid, is, as I said, in a duplicate ratio of the velocity; the other part, which arises from the tenacity of the fluid, is uniform, or as the moment of the time; and, therefore, we might now proceed to the motion of bodies, which are resisted partly by an uniform force, or in the ratio of the moments of the time, and partly in the duplicate ratio of the velocity. But it is sufficient to have cleared the way to this speculation in the 8th and 9th prop. foregoing, and their corollaries. For in those propositions, instead of the uniform resistance made to an ascending body arising from its gravity, one may substitute the uniform resistance which arises from the tenacity of the medium, when the body moves by its *vis insita* alone; and when the body ascends in a right line, add this uniform resistance to the force of gravity, and subduct it when the body descends in a right line. One might also go on to the motion of bodies which are resisted in part uniformly, in part in the ratio of the velocity, and in part in the duplicate ratio of the same velo-

city. And I have opened a way to this in the 13th and 14th prop. foregoing, in which the uniform resistance arising from the tenacity of the medium may be substituted for the force of gravity, or be compounded with it as before. But I hasten to other things.

SECTION IV.

Of the circular motion of bodies in resisting mediums.

LEMMA III.

Let PQR be a spiral, cutting all the radii $SP, SQ, SR, \&c.$ in equal angles. Draw the right line PT touching the spiral in any point P , and cutting the radius SQ in T ; draw PO, QO perpendicular to the spiral, and meeting in O , and join SO . I say, that if the points P and Q approach and coincide, the angle PSO will become a right angle, and the ultimate ratio of the rectangle $TQ \times 2PS$ to PQ^2 will be the ratio of equality. (Pl. 4, Fig. 1.)

For, from the right angles OPQ, OQR , subduct the equal angles SPQ, SQR , and there will remain the equal angles OPS, OQS . Therefore a circle which passes through the points OSP will pass also through the point Q . Let the points P and Q coincide, and this circle will touch the spiral in the place of coincidence PQ , and will therefore cut the right line OP perpendicularly. Therefore OP will become a diameter of this circle, and the angle OSP , being in a semicircle, becomes a right one. Q.E.D.

Draw QD, SE perpendicular to OP , and the ultimate ratios of the lines will be as follow: TQ to PD as TS or PS to PE , or $2PO$ to $2PS$; and PD to PQ as PQ to $2PO$; and, *ex æquo perturbatè*, TQ to PQ as PQ to $2PS$. Whence PQ^2 becomes equal to $TQ \times 2PS$. Q.E.D.

PROPOSITION XV. THEOREM XII.

If the density of a medium in each place thereof be reciprocally as the distance of the places from an immovable centre, and the centripetal force be in the duplicate ratio of the density; I say, that a body may revolve in a spiral which cuts all the radii drawn from that centre in a given angle. (Pl. 4, Fig. 2.)

Suppose every thing to be as in the foregoing lemma, and produce SQ to V, so that SV may be equal to SP. In any time let a body, in a resisting medium, describe the least arc PQ, and in double the time the least arc PR; and the decrements of those arcs arising from the resistance, or their differences from the arcs which would be described in a non-resisting medium in the same times, will be to each other as the squares of the times in which they are generated; therefore the decrement of the arc PQ is the fourth part of the decrement of the arc PR. Whence also if the area QSr be taken equal to the area PSQ, the decrement of the arc PQ will be equal to half the lineolæ Rr; and therefore the force of resistance and the centripetal force are to each other as the lineolæ $\frac{1}{2}R$ and TQ, which they generate in the same time. Because the centripetal force with which the body is urged in P is reciprocally as SP^2 , and (by lem. 10, book 1) the lineolæ TQ, which is generated by that force, is in a ratio compounded of the ratio of this force and the duplicate ratio of the time in which the arc PQ is described (for in this case I neglect the resistance, as being infinitely less than the centripetal force), it follows that $TQ \times SP^2$, that is (by the last lemma), $\frac{1}{2}PQ^2 \times SP$, will be in a duplicate ratio of the time, and therefore the time is as $PQ \times \sqrt{SP}$; and the velocity of the body, with which the arc PQ is described in that time, as $\frac{PQ}{PQ \times \sqrt{SP}}$ or $\frac{1}{\sqrt{SP}}$, that is, in the subduplicate ratio of SP reciprocally. And, by a like reasoning, the velocity with which the arc QR is described, is in the subduplicate ratio of SQ reciprocally. Now those arcs PQ and QR are as the describing velocities to each other; that is, in the subduplicate ratio of SQ to SP, or as SQ to $\sqrt{SP \times SQ}$; and, because of the equal angles SPQ, SQR, and the equal areas PSQ, QSr, the arc PQ is to the arc QR as SQ to SP. Take the differences of the proportional consequents, and the arc PQ will be to the arc Rr as SQ to $SP - \sqrt{SP \times SQ}$, or $\frac{1}{2}VQ$. For the points P and Q coinciding, the ultimate ratio of $SP - \sqrt{SP \times SQ}$ to $\frac{1}{2}VQ$ is the ratio of equality. Because the decrement of the arc PQ arising from the resistance, or its

double Rr , is as the resistance and the square of the time conjunctly, the resistance will be as $\frac{Rr}{PQ^2 \times SP}$. But PQ was

to Rr as SQ to $\frac{1}{2}VQ$, and thence $\frac{Rr}{PQ^2 \times SP}$ becomes as

$\frac{\frac{1}{2}VQ}{PQ \times SP \times SQ}$, or as $\frac{\frac{1}{2}OS}{OP \times SP^2}$. For the points P and Q

coinciding, SP and SQ coincide also, and the angle PVQ becomes a right one; and, because of the similar triangles PVQ , PSO , PQ becomes to $\frac{1}{2}VQ$ as OP to $\frac{1}{2}OS$. Therefore

$\frac{OS}{OP \times SP^2}$ is as the resistance, that is, in the ratio of the density of the medium in P and the duplicate ratio of the velocity conjunctly. Subtract the duplicate ratio of the velocity, namely, the ratio $\frac{1}{SP}$, and there will remain the density of

the medium in P , as $\frac{OS}{OP \times SP}$. Let the spiral be given,

and, because of the given ratio of OS to OP , the density of the medium in P will be as $\frac{1}{SP}$. Therefore in a medium whose

density is reciprocally as SP the distance from the centre, a body will revolve in this spiral. Q.E.D.

COR. 1. The velocity in any place P is always the same wherewith a body in a non-resisting medium with the same centripetal force would revolve in a circle, at the same distance SP from the centre.

COR. 2. The density of the medium, if the distance SP be given, is as $\frac{OS}{OP}$, but if that distance is not given, as $\frac{OS}{OP \times SP}$.

And thence a spiral may be fitted to any density of the medium.

COR. 3. The force of the resistance in any place P is to the centripetal force in the same place as $\frac{1}{2}OS$ to OP . For those forces are to each other as $\frac{1}{2}Rr$ and TQ , or as $\frac{\frac{1}{2}VQ \times PQ}{SQ}$

and $\frac{1}{2} \frac{PQ^2}{SP}$, that is, as $\frac{1}{2} VQ$ and PQ , or $\frac{1}{2} OS$ and OP . The spiral, therefore, being given, there is given the proportion of the resistance to the centripetal force; and, *vice versa*, from that proportion given the spiral is given.

COR. 4. Therefore the body cannot revolve in this spiral, except where the force of resistance is less than half the centripetal force. Let the resistance be made equal to half the centripetal force, and the spiral will coincide with the right line PS , and in that right line the body will descend to the centre with a velocity that is to the velocity, with which it was proved before, in the case of the parabola (theor. 10, book 1), the descent would be made in a non-resisting medium, in the subduplicate ratio of unity to the number two. And the times of the descent will be here reciprocally as the velocities, and therefore given.

COR. 5. And because at equal distances from the centre the velocity is the same in the spiral PQR as it is in the right line SP , and the length of the spiral is to the length of the right line PS in a given ratio, namely, in the ratio of OP to OS ; the time of the descent in the spiral will be to the time of the descent in the right line SP in the same given ratio, and therefore given.

COR. 6. If from the centre S , with any two given intervals, two circles are described; and, these circles remaining, the angle which the spiral makes with the radius PS be any how changed; the number of revolutions which the body can complete in the space between the circumferences of those circles, going round in the spiral from one circumference to another, will be as $\frac{PS}{OS}$, or as the tangent of the angle which the spiral makes with the radius PS ; and the time of the same revolutions will be as $\frac{OP}{OS}$, that is, as the secant of the same angle, or reciprocally as the density of the medium.

COR. 7. If a body, in a medium whose density is reciprocally as the distances of places from the centre, revolves in any

curve AEB (Pl. 4, Fig. 3) about that centre, and cuts the first radius AS in the same angle in B as it did before in A, and that with a velocity that shall be to its first velocity in A reciprocally in a subduplicate ratio of the distances from the centre (that is, as AS to a mean proportional between AS and BS), that body will continue to describe innumerable similar revolutions BFC, CGD, &c., and by its interfections will distinguish the radius AS into parts AS, BS, CS, DS, &c. that are continually proportional. But the times of the revolutions will be as the perimeters of the orbits AEB, BFC, CGD, &c. directly, and the velocities at the beginnings A, B, C of those orbits inversely; that is, as $AS^{\frac{3}{2}}$, $BS^{\frac{3}{2}}$, $CS^{\frac{3}{2}}$. And the whole time in which the body will arrive at the centre, will be to the time of the first revolution as the sum of all the continued proportionals $AS^{\frac{3}{2}}$, $BS^{\frac{3}{2}}$, $CS^{\frac{3}{2}}$, going on *ad infinitum*, to the first term $AS^{\frac{3}{2}}$; that is, as the first term $AS^{\frac{3}{2}}$ to the difference of the two first $AS^{\frac{3}{2}} - BS^{\frac{3}{2}}$, or as $\frac{2}{3}AS$ to AB very nearly. Whence the whole time may be easily found.

COR. 8. From hence also may be deduced, near enough, the motions of bodies in mediums whose density is either uniform, or observes any other assigned law. From the centre S, with intervals SA, SB, SC, &c. continually proportional, describe as many circles; and suppose the time of the revolutions between the perimeters of any two of those circles, in the medium whereof we treated, to be to the time of the revolutions between the same in the medium proposed as the mean density of the proposed medium between those circles to the mean density of the medium whereof we treated, between the same circles, nearly; and that the secant of the angle in which the spiral above determined, in the medium whereof we treated, cuts the radius AS, is in the same ratio to the secant of the angle in which the new spiral, in the proposed medium, cuts the same radius: and also that the number of all the revolutions between the same two circles is nearly as the tangents of those angles. If this be done every where

between every two circles, the motion will be continued through all the circles. And by this means one may, without difficulty, conceive at what rate and in what time bodies ought to revolve in any regular medium.

COR. 9. And although these motions, becoming eccentric, should be performed in spirals approaching to an oval figure, yet, conceiving the several revolutions of those spirals to be at the same distances from each other, and to approach to the centre by the same degrees as the spiral above described, we may also understand how the motions of bodies may be performed in spirals of that kind.

PROPOSITION XVI. THEOREM XIII.

If the density of the medium in each of the places be reciprocally as the distance of the places from the immovable centre, and the centripetal force be reciprocally as any power of the same distance, I say, that the body may revolve in a spiral, intersecting all the radii drawn from that centre in a given angle. (Pl. 4, Fig. 2.)

This is demonstrated in the same manner as the foregoing proposition. For, if the centripetal force in P be reciprocally as any power SP^{n+1} of the distance SP, whose index is $n+1$, it will be collected, as above, that the time in which the body describes any arc PQ, will be as $PQ \times PS^{\frac{1}{2}n}$; and the resistance in P as $\frac{Rr}{PQ^2 \times SP^n}$, or as $\frac{1 - \frac{1}{2}n \times VQ}{PQ \times SP^n \times SQ}$, and therefore as $\frac{1 - \frac{1}{2}n \times OS}{OP \times SP^{n+1}}$, that is, (because $\frac{1 - \frac{1}{2}n \times OS}{OP}$ is a given quantity), reciprocally as SP^{n+1} . And therefore, since the velocity is reciprocally as $SP^{\frac{1}{2}n}$, the density in P will be reciprocally as SP.

COR. 1. The resistance is to the centripetal force as $1 - \frac{1}{2}n \times OS$ to OP.

COR. 2. If the centripetal force be reciprocally as SP^3 , $1 - \frac{1}{2}n$ will be $= 0$; and therefore the resistance and density of the medium will be nothing, as in prop. 9, book 1.

COR. 3. If the centripetal force be reciprocally as any power of the radius SP, whose index is greater than the num-

ber 3, the affirmative resistance will be changed into a negative.

SCHOLIUM.

This proposition and the former, which relate to mediums of unequal density, are to be understood of the motion of bodies that are so small, that the greater density of the medium on one side of the body above that on the other is not to be considered. I suppose also the resistance, *cæteris paribus*, to be proportional to its density. Whence, in mediums whose force of resistance is not as the density, the density must be so much augmented or diminished, that either the excess of the resistance may be taken away, or the defect supplied.

PROPOSITION XVII. PROBLEM IV.

To find the centripetal force and the resisting force of the medium, by which a body, the law of the velocity being given, shall revolve in a given spiral. (Pl. 4, Fig. 4.)

Let that spiral be PQR. From the velocity, with which the body goes over the very small arc PQ, the time will be given; and from the altitude TQ, which is as the centripetal force, and the square of the time, that force will be given. Then from the difference RSr of the areas PSQ and QSR described in equal particles of time, the retardation of the body will be given; and from the retardation will be found the resisting force and density of the medium.

PROPOSITION XVIII. PROBLEM V.

The law of centripetal force being given, to find the density of the medium in each of the places thereof, by which a body may describe a given spiral.

From the centripetal force the velocity in each place must be found; then from the retardation of the velocity the density of the medium is found, as in the foregoing proposition.

But I have explained the method of managing these problems in the tenth proposition and second lemma of this book; and will no longer detain the reader in these perplexed disquisitions. I shall now add some things relating to the forces of progressive bodies, and to the density and resistance of those mediums in which the motions hitherto treated of, and those akin to them, are performed.

SECTION V.

Of the density and compression of fluids; and of hydrostatics.

THE DEFINITION OF A FLUID.

A fluid is any body whose parts yield to any force impressed on it, and, by yielding, are easily moved among themselves.

PROPOSITION XIX. THEOREM XIV.

All the parts of a homogeneous and unmoved fluid included in any unmoved vessel, and compressed on every side (setting aside the consideration of condensation, gravity, and all centripetal forces), will be equally pressed on every side, and remain in their places without any motion arising from that pressure. (Pl. 4, Fig. 5.)

CASE 1. Let a fluid be included in a spherical vessel ABC, and uniformly compressed on every side: I say, that no part of it will be moved by that pressure. For if any part, as D, be moved, all such parts at the same distance from the centre on every side must necessarily be moved at the same time by a like motion; because the pressure of them all is similar and equal; and all other motion is excluded that does not arise from that pressure. But if these parts come all of them nearer to the centre, the fluid must be condensed towards the centre, contrary to the supposition. If they recede from it, the fluid must be condensed towards the circumference; which is also contrary to the supposition. Neither can they move in any one direction retaining their distance from the centre, because, for the same reason, they may move in a contrary direction; but the same part cannot be moved contrary ways at the same time. Therefore no part of the fluid will be moved from its place. Q.E.D.

CASE 2. I say now, that all the spherical parts of this fluid are equally pressed on every side. For let EF be a spherical part of the fluid; if this be not pressed equally on every side, augment the lesser pressure till it be pressed equally on every side; and its parts (by case 1) will remain in their places. But before the increase of the pressure, they would remain in their places (by case 1); and by

the addition of a new pressure they will be moved, by the definition of a fluid, from those places. Now these two conclusions contradict each other. Therefore it was false to say that the sphere EF was not pressed equally on every side. Q.E.D.

CASE 3. I say besides, that different spherical parts have equal pressures. For the contiguous spherical parts press each other mutually and equally in the point of contact (by law 3). But (by case 2) they are pressed on every side with the same force. Therefore any two spherical parts not contiguous, since an intermediate spherical part can touch both, will be pressed with the same force. Q.E.D.

CASE 4. I say now, that all the parts of the fluid are every where pressed equally. For any two parts may be touched by spherical parts in any points whatever; and there they will equally press those spherical parts (by case 3), and are reciprocally equally pressed by them (by law 3). Q.E.D.

CASE 5. Since, therefore, any part GHI of the fluid is inclosed by the rest of the fluid as in a vessel, and is equally pressed on every side; and also its parts equally press one another, and are at rest among themselves; it is manifest that all the parts of any fluid as GHI, which is pressed equally on every side, do press each other mutually and equally, and are at rest among themselves. Q.E.D.

CASE 6. Therefore if that fluid be included in a vessel of a yielding substance, or that is not rigid, and be not equally pressed on every side, the same will give way to a stronger pressure, by the definition of fluidity.

CASE 7. And therefore, in an inflexible or rigid vessel, a fluid will not sustain a stronger pressure on one side than on the other, but will give way to it, and that in a moment of time; because the rigid side of the vessel does not follow the yielding liquor. But the fluid, by thus yielding, will press against the opposite side, and so the pressure will tend on every side to equality. And because the fluid, as soon as it endeavours to recede from the part that is most pressed, is withstood by the resistance of the vessel on the opposite side, the pressure will

on every side be reduced to equality, in a moment of time, without any local motion: and from thence the parts of the fluid (by case 5) will press each other mutually and equally, and be at rest among themselves. Q.E.D.

COR. Whence neither will a motion of the parts of the fluid among themselves be changed by a pressure communicated to the external superficies, except so far as either the figure of the superficies may be somewhere altered, or that all the parts of the fluid, by pressing one another more intensely or remissly, may slide with more or less difficulty among themselves.

PROPOSITION XX. THEOREM XV.

If all the parts of a spherical fluid, homogeneous at equal distances from the centre, lying on a spherical concentric bottom, gravitate towards the centre of the whole, the bottom will sustain the weight of a cylinder, whose base is equal to the superficies of the bottom, and whose altitude is the same with that of the incumbent fluid. (Pl. 4, Fig. 6.)

Let DHM be the superficies of the bottom, and AEI the upper superficies of the fluid. Let the fluid be distinguished into concentric orbs of equal thickness, by the innumerable spherical superficies BFK, CGL; and conceive the force of gravity to act only in the upper superficies of every orb, and the actions to be equal on the equal parts of all the superficies. Therefore the upper superficies AE is pressed by the single force of its own gravity, by which all the parts of the upper orb, and the second superficies BFK, will (by prop. 19), according to its measure, be equally pressed. The second superficies BFK is pressed likewise by the force of its own gravity, which, added to the former force, makes the pressure double. The third superficies CGL is, according to its measure, acted on by this pressure and the force of its own gravity besides, which makes its pressure triple. And in like manner the fourth superficies receives a quadruple pressure, the fifth superficies a quintuple, and so on. Therefore the pressure acting on every superficies is not as the solid quantity of the incumbent fluid, but as the number of the orbs reaching to the upper surface of the fluid; and is equal to the

gravity of the lowest orb multiplied by the number of orbs : that is, to the gravity of a solid whose ultimate ratio to the cylinder abovementioned (when the number of the orbs is increased and their thickness diminished, *ad infinitum*, so that the action of gravity from the lowest superficies to the uppermost may become continued) is the ratio of equality. Therefore the lowest superficies sustains the weight of the cylinder above determined. Q.E.D. And, by a like reasoning, the proposition will be evident, where the gravity of the fluid decreases in any assigned ratio of the distance from the centre, and also where the fluid is more rare above and denser below. Q.E.D.

COR. 1. Therefore the bottom is not pressed by the whole weight of the incumbent fluid, but only sustains that part of it which is described in the proposition ; the rest of the weight being sustained archwise by the spherical figure of the fluid.

COR. 2. The quantity of the pressure is the same always at equal distances from the centre, whether the superficies pressed be parallel to the horizon, or perpendicular, or oblique ; or whether the fluid, continued upwards from the compressed superficies, rises perpendicularly in a rectilinear direction, or creeps obliquely through crooked cavities and canals, whether those passages be regular or irregular, wide or narrow. That the pressure is not altered by any of these circumstances, may be collected by applying the demonstration of this theorem to the several cases of fluids.

COR. 3. From the same demonstration it may also be collected (by prop. 19), that the parts of a heavy fluid acquire no motion among themselves by the pressure of the incumbent weight, except that motion which arises from condensation.

COR. 4. And therefore if another body of the same specific gravity, incapable of condensation, be immersed in this fluid, it will acquire no motion by the pressure of the incumbent weight : it will neither descend nor ascend, nor change its figure. If it be spherical, it will remain so, notwithstanding the pressure ; if it be square, it will remain square ; and that, whether it be soft or fluid ; whether it swims freely in the

fluid, or lies at the bottom. For any internal part of a fluid is in the same state with the submersed body; and the case of all submersed bodies that have the same magnitude, figure, and specific gravity, is alike. If a submersed body, retaining its weight, should dissolve and put on the form of a fluid, this body, if before it would have ascended, descended, or from any pressure assume a new figure, would now likewise ascend, descend, or put on a new figure; and that, because its gravity and the other causes of its motion remain. But (by case 5, prop. 19) it would now be at rest, and retain its figure. Therefore also in the former case.

COR. 5. Therefore a body that is specifically heavier than a fluid contiguous to it will sink; and that which is specifically lighter will ascend, and attain so much motion and change of figure as that excess or defect of gravity is able to produce; for that excess or defect is the same thing as an impulse, by which a body, otherwise *in equilibrio* with the parts of the fluid, is acted on; and may be compared with the excess or defect of a weight in one of the scales of a balance.

COR. 6. Therefore bodies placed in fluids have a twofold gravity; the one true and absolute, the other apparent, vulgar, and comparative. Absolute gravity is the whole force with which the body tends downwards; relative and vulgar gravity is the excess of gravity with which the body tends downwards more than the ambient fluid. By the first kind of gravity the parts of all fluids and bodies gravitate in their proper places; and therefore their weights taken together compose the weight of the whole. For the whole taken together is heavy, as may be experienced in vessels full of liquor; and the weight of the whole is equal to the weights of all the parts, and is, therefore, composed of them. By the other kind of gravity bodies do not gravitate in their places; that is, compared with one another, they do not preponderate, but, hindering one another's endeavours to descend, remain in their proper places, as if they were not heavy. Those things which are in the air, and do not preponderate, are commonly looked on as not heavy. Those which do preponderate are commonly reckoned heavy, inasmuch as they are not sustained by

the weight of the air. The common weights are nothing else but the excess of the true weights above the weight of the air. Hence also, vulgarly, those things are called light which are less heavy, and, by yielding to the preponderating air, mount upwards. But these are only comparatively light, and not truly so, because they descend *in vacuo*. Thus, in water, bodies which, by their greater or less gravity, descend or ascend, are comparatively and apparently heavy or light; and their comparative and apparent gravity or levity is the excess or defect by which their true gravity either exceeds the gravity of the water or is exceeded by it. But those things which neither by preponderating descend, nor, by yielding to the preponderating fluid, ascend, although by their true weight they do increase the weight of the whole, yet comparatively, and in the sense of the vulgar, they do not gravitate in the water. For these cases are alike demonstrated.

COR. 7. These things, which have been demonstrated concerning gravity, take place in any other centripetal forces.

COR. 8. Therefore, if the medium in which any body moves be acted on either by its own gravity, or by any other centripetal force, and the body be urged more powerfully by the same force; the difference of the forces is that very motive force, which, in the foregoing propositions, I have considered as a centripetal force. But, if the body be more lightly urged by that force, the difference of the forces becomes a centrifugal force, and is to be considered as such.

COR. 9. But since fluids, by pressing the included bodies, do not change their external figures, it appears also (by cor. prop. 19) that they will not change the situation of their internal parts in relation to one another; and therefore, if animals were immersed therein, and that all sensation did arise from the motion of their parts, the fluid will neither hurt the immersed bodies, nor excite any sensation, unless so far as those bodies may be condensed by the compression. And the case is the same of any system of bodies encompassed with a compressing fluid. All the parts of the system will be agitated with the same motions as if they were placed in a vacuum, and would only retain their comparative gravity;

unless so far as the fluid may somewhat resist their motions, or be requisite to conglutinate them by compression.

PROPOSITION XXI. THEOREM XVI.

Let the density of any fluid be proportional to the compression, and its parts be attracted downwards by a centripetal force reciprocally proportional to the distances from the centre: I say, that, if those distances be taken continually proportional, the densities of the fluid at the same distances will be also continually proportional. (Pl. 5, Fig. 1.)

Let ATV denote the spherical bottom of the fluid, S the centre, SA, SB, SC, SD, SE, SF, &c. distances continually proportional. Erect the perpendiculars AH, BI, CK, DL, EM, FN, &c. which shall be as the densities of the medium in the places A, B, C, D, E, F; and the specific gravities in those places will be as $\frac{AH}{AS}, \frac{BI}{BS}, \frac{CK}{CS},$ &c. or, which is all one, as $\frac{AH}{AB}, \frac{BI}{BC}, \frac{CK}{CD},$ &c. Suppose, first, these gravities

to be uniformly continued from A to B, from B to C, from C to D, &c. the decrements in the points B, C, D, &c. being taken by steps. And these gravities, drawn into the altitudes AB, BC, CD, &c. will give the pressures AH, BI, CK, &c. by which the bottom ATV is acted on (by theor. 15). Therefore the particle A sustains all the pressures AH, BI, CK, DL, &c. proceeding *in infinitum*; and the particle B sustains the pressures of all but the first, AH; and the particle C all but the two first, AH, BI; and so on: and therefore the density AH of the first particle A is to the density BI of the second particle B as the sum of all AH + BI + CK + DL, *in infinitum*, to the sum of all BI + CK + DL, &c. And BI the density of the second particle B is to CK the density of the third C, as the sum of all BI + CK + DL, &c. to the sum of all CK + DL, &c. Therefore these sums are proportional to their differences AH, BI, CK, &c. and therefore continually proportional (by lem. 1 of this book); and therefore the differences AH, BI, CK, &c. proportional to the sums, are also continually proportional. Wherefore, since the densities in the places A, B, C, &c. are as AH, BI, CK, &c.

they will also be continually proportional. Proceed intermissively, and, *ex æquo*, at the distances SA, SC, SE, continually proportional, the densities AH, CK, EM will be continually proportional. And, by the same reasoning, at any distances SA, SD, SG, continually proportional, the densities AH, DL, GO, will be continually proportional. Let now the points A, B, C, D, E, &c. coincide, so that the progression of the specific gravities from the bottom A to the top of the fluid may be made continual; and at any distances SA, SD, SG, continually proportional, the densities AH, DL, GO, being all along continually proportional, will still remain continually proportional. Q.E.D.

COR. Hence, if the density of the fluid in two places, as A and E, be given, its density in any other place Q may be collected. With the centre S, and the rectangular asymptotes SQ, SX, describe (Fig. 2) an hyperbola cutting the perpendiculars AH, EM, QT in a, e, and q, as also the perpendiculars HX, MY, TZ, let fall upon the asymptote SX, in h, m, and t. Make the area YmtZ to the given area YmhX as the given area EeqQ to the given area EeaA; and the line Zt produced will cut off the line QT proportional to the density. For, if the lines SA, SE, SQ, are continually proportional, the areas EeqQ, EeaA will be equal, and thence the areas YmtZ, Xhmy, proportional to them, will be also equal; and the lines SX, SY, SZ, that is, AH, EM, QT continually proportional, as they ought to be. And, if the lines SA, SE, SQ, obtain any other order in the series of continued proportionals, the lines AH, EM, QT, because of the proportional hyperbolic areas, will obtain the same order in another series of quantities continually proportional.

PROPOSITION XXII. THEOREM XVII.

Let the density of any fluid be proportional to the compression, and its parts be attracted downwards by a gravitation reciprocally proportional to the squares of the distances from the centre: I say, that, if the distances be taken in harmonic progression, the densities of the fluid at those distances will be in a geometrical progression. (Pl. 5, Fig. 3.)

Let S denote the centre, and SA, SB, SC, SD, SE, the distances in geometrical progression. Erect the perpendiculars AH,

BI, CK, &c. which shall be as the densities of the fluid in the places A, B, C, D, E, &c. and the specific gravities thereof in those places will be as $\frac{AH}{SA^2}, \frac{BI}{SB^2}, \frac{CK}{SC^2},$ &c. Suppose these gravities to be uniformly continued, the first from A to B, the second from B to C, the third from C to D, &c. And these, drawn into the altitudes AB, BC, CD, DE, &c., or, which is the same thing, into the distances SA, SB, SC, &c. proportional to those altitudes, will give $\frac{AH}{SA}, \frac{BI}{SB}, \frac{CK}{SC},$ &c. the exponents of the pressures. Therefore, since the densities are as the sums of those pressures, the differences AH — BI, BI — CK, &c. of the densities will be as the differences of those sums $\frac{AH}{SA}, \frac{BI}{SB}, \frac{CK}{SC},$ &c. With the centre S, and the asymptotes SA, Sx, describe any hyperbola, cutting the perpendiculars AH, BI, CK, &c. in a, b, c, &c. and the perpendiculars Ht, Iu, Kw, let fall upon the asymptote Sx, in h, i, k; and the differences of the densities tu, uw, &c. will be as $\frac{AH}{SA}, \frac{BI}{SB},$ &c. And the rectangles $tu \times th, uw \times ui,$ &c. or tp, uq, &c. as $\frac{AH \times th}{SA}, \frac{BI \times ui}{SB},$ &c. that is, as Aa, Bb, &c. For, by the nature of the hyperbola, SA is to AH or St as th to Aa, and therefore $\frac{AH \times th}{SA}$ is equal to Aa. And, by a like reasoning, $\frac{BI \times ui}{SB}$ is equal to Bb, &c. But Aa, Bb, Cc, &c. are continually proportional, and therefore proportional to their differences Aa — Bb, Bb — Cc, &c., and therefore the rectangles tp, uq, &c. are proportional to those differences; as also the sums of the rectangles $tp + uq,$ or $tp + uq + wr,$ to the sums of the differences Aa — Cc or Aa — Dd. Suppose several of these terms, and the sum of all the differences, as Aa — Ff, will be proportional to the sum of all the rectangles, as zthn. Increase the number of terms, and diminish the distances of the points A, B, C, &c. in infinitum,

and those rectangles will become equal to the hyperbolic area $zthn$, and therefore the difference $Aa - Ff$ is proportional to this area. Take now any distances, as SA, SD, SF , in harmonic progression, and the differences $Aa - Dd, Dd - Ff$ will be equal; and therefore the areas $thlx, xlnz$, proportional to those differences, will be equal among themselves, and the densities St, Sx, Sz , that is, AH, DL, FN , continually proportional. Q.E.D.

COR. 2. Hence if any two densities of the fluid, as AH and Bl , be given, the area $thiu$, answering to their difference tu , will be given; and thence the density FN will be found at any height SF , by taking the area $thnz$ to that given area $thiu$ as the difference $Aa - Ff$ to the difference $Aa - Bb$.

SCHOLIUM.

By a like reasoning it may be proved, that if the gravity of the particles of a fluid be diminished in a triplicate ratio of the distance from the centre; and the reciprocals of the squares of the distances $SA, SB, SC, \&c.$ (namely, $\frac{SA^3}{SA^2}, \frac{SA^3}{SB^2}, \frac{SA^3}{SC^2}$), be taken in an arithmetical progression, the densities $AH, BI, CK, \&c.$ will be in a geometrical progression. And if the gravity be diminished in a quadruplicate ratio of the distances, and the reciprocals of the cubes of the distances (as $\frac{SA^4}{SA^3}, \frac{SA^4}{SB^3}, \frac{SA^4}{SC^3}, \&c.$) be taken in arithmetical progression, the densities $AH, BI, CK, \&c.$ will be in geometrical progression. And so *in infinitum*. Again; if the gravity of the particles of the fluid be the same at all distances, and the distances be in arithmetical progression, the densities will be in a geometrical progression, as Dr. *Halley* has found. If the gravity be as the distance, and the squares of the distances be in arithmetical progression, the densities will be in geometrical progression. And so *in infinitum*. These things will be so, when the density of the fluid condensed by compression is as the force of compression; or, which is the same thing, when the space possessed by the fluid is reciprocally as this force. Other laws of condensation may be supposed, as that the cube of the compressing force may be as the biqua-

drate of the density ; or the triplicate ratio of the force the same with the quadruplicate ratio of the density : in which case, if the gravity be reciprocally as the square of the distance from the centre, the density will be reciprocally as the cube of the distance. Suppose that the cube of the compressing force be as the quadrato-cube of the density ; and if the gravity be reciprocally as the square of the distance, the density will be reciprocally in a sesquiplicate ratio of the distance. Suppose the compressing force to be in a duplicate ratio of the density, and the gravity reciprocally in a duplicate ratio of the distance, and the density will be reciprocally as the distance. To run over all the cases that might be offered would be tedious. But as to our own air, this is certain from experiment, that its density is either accurately, or very nearly at least, as the compressing force ; and therefore the density of the air in the atmosphere of the earth is as the weight of the whole incumbent air, that is, as the height of the mercury in the barometer.

PROPOSITION XXIII. THEOREM XVIII.

If a fluid be composed of particles mutually flying each other, and the density be as the compression, the centrifugal forces of the particles will be reciprocally proportional to the distances of their centres. And, vice versa, particles flying each other with forces that are reciprocally proportional to the distances of their centres, compose an elastic fluid, whose density is as the compression. (Pl. 5, Fig. 4.)

Let the fluid be supposed to be included in a cubic space ACE, and then to be reduced by compression into a lesser cubic space ace ; and the distances of the particles retaining a like situation with respect to each other in both the spaces will be as the sides AB, ab, of the cubes ; and the densities of the mediums will be reciprocally as the containing spaces AB^3 , ab^3 . In the plane side of the greater cube ABCD take the square DP equal to the plane side db of the lesser cube ; and, by the supposition, the pressure with which the square DP urges the inclosed fluid will be to the pressure with which that square db urges the inclosed fluid as the densities of the mediums are to each other, that is, as ab^3 to AB^3 . But the

pressure with which the square DB urges the included fluid is to the pressure with which the square DP urges the same fluid as the square DB to the square DP, that is, as AB^2 to ab^2 . Therefore, *ex æquo*, the pressure with which the square DB urges the fluid is to the pressure with which the square db urges the fluid as ab to AB . Let the planes FGH, fgh, be drawn through the middles of the two cubes, and divide the fluid into two parts. These parts will press each other mutually with the same forces with which they are themselves pressed by the planes AC, ac, that is, in the proportion of ab to AB ; and therefore the centrifugal forces by which these pressures are sustained are in the same ratio. The number of the particles being equal, and the situation alike, in both cubes, the forces which all the particles exert, according to the planes FGH, fgh, upon all, are as the forces which each exerts on each. Therefore the forces which each exerts on each, according to the plane FGH in the greater cube, are to the forces which each exerts on each, according to the plane fgh in the lesser cube, as ab to AB , that is, reciprocally as the distance of the particles from each other. Q.E.D.

And, *vice versa*, if the forces of the single particles are reciprocally as the distances, that is, reciprocally as the sides of the cubes AB , ab ; the sums of the forces will be in the same ratio, and the pressure of the sides DB, db, as the sums of the forces; and the pressure of the square DP to the pressure of the side DB as ab^2 to AB^2 . And, *ex æquo*, the pressure of the square DP to the pressure of the side db as ab^3 to AB^3 ; that is, the force of compression in the one to the force of compression in the other as the density in the former to the density in the latter. Q.E.D.

SCHOLIUM.

By a like reasoning, if the centrifugal forces of the particles are reciprocally in the duplicate ratio of the distances between the centres, the cubes of the compressing forces will be as the biquadrates of the densities. If the centrifugal forces be reciprocally in the triplicate or quadruplicate ratio of the distances, the cubes of the compressing forces will be as the quadrato-cubes or cubo-cubes of the densities. And univer-

sally, if D be put for the distance, and E for the density of the compressed fluid, and the centrifugal forces be reciprocally as any power D^n of the distance, whose index is the number n , the compressing forces will be as the cube roots of the power $E^n + 2$, whose index is the number $n + 2$: and the contrary. All these things are to be understood of particles whose centrifugal forces terminate in those particles that are next them, or are diffused not much further. We have an example of this in magnetical bodies. Their attractive virtue is terminated nearly in bodies of their own kind that are next them. The virtue of the magnet is contracted by the interposition of an iron plate, and is almost terminated at it; for bodies further off are not attracted by the magnet so much as by the iron plate. If in this manner particles repel others of their own kind that lie next them, but do not exert their virtue on the more remote, particles of this kind will compose such fluids as are treated of in this proposition. If the virtue of any particle diffuse itself every way *in infinitum*, there will be required a greater force to produce an equal condensation of a greater quantity of the fluid. But whether elastic fluids do really consist of particles so repelling each other, is a physical question. We have here demonstrated mathematically the property of fluids consisting of particles of this kind, that hence philosophers may take occasion to discuss that question.

SECTION VI.

Of the motion and resistance of funependulous bodies.

PROPOSITION XXIV. THEOREM XIX.

The quantities of matter in funependulous bodies, whose centres of oscillation are equally distant from the centre of suspension, are in a ratio compounded of the ratio of the weights and the duplicate ratio of the times of the oscillations in vacuo.

For the velocity which a given force can generate in a given matter in a given time is as the force and the time directly, and the matter inversely. The greater the force or the time is, or the less the matter, the greater velocity will be generated. This is manifest from the second law of motion. Now if pendulums are of the same length, the motive forces

in places equally distant from the perpendicular are as the weights : and therefore if two bodies by oscillating describe equal arcs, and those arcs are divided into equal parts ; since the times in which the bodies describe each of the correspondent parts of the arcs are as the times of the whole oscillations, the velocities in the correspondent parts of the oscillations will be to each other as the motive forces and the whole times of the oscillations directly, and the quantities of matter reciprocally : and therefore the quantities of matter are as the forces and the times of the oscillations directly and the velocities reciprocally. But the velocities reciprocally are as the times, and therefore the times directly and the velocities reciprocally are as the squares of the times ; and therefore the quantities of matter are as the motive forces and the squares of the times, that is, as the weights and the squares of the times. Q.E.D.

COR. 1. Therefore if the times are equal, the quantities of matter in each of the bodies are as the weights.

COR. 2. If the weights are equal, the quantities of matter will be as the squares of the times.

COR. 3. If the quantities of matter are equal, the weights will be reciprocally as the squares of the times.

COR. 4. Whence since the squares of the times, *cæteris paribus*, are as the lengths of the pendulums, therefore if both the times and quantities of matter are equal, the weights will be as the lengths of the pendulums.

COR. 5. And universally, the quantity of matter in the pendulous body is as the weight and the square of the time directly, and the length of the pendulum inversely.

COR. 6. But in a non-resisting medium, the quantity of matter in the pendulous body is as the comparative weight and the square of the time directly, and the length of the pendulum inversely. For the comparative weight is the motive force of the body in any heavy medium, as was shewn above ; and therefore does the same thing in such a non-resisting medium as the absolute weight does in a vacuum.

COR. 7. And hence appears a method both of comparing bodies one among another, as to the quantity of matter in

each; and of comparing the weights of the same body in different places, to know the variation of its gravity. And by experiments made with the greatest accuracy, I have always found the quantity of matter in bodies to be proportional to their weight.

PROPOSITION XXV. THEOREM XX.

Funependulous bodies that are, in any medium, resisted in the ratio of the moments of time, and funependulous bodies that move in a non-resisting medium of the same specific gravity, perform their oscillations in a cycloid in the same time, and describe proportional parts of arcs together. (Pl. 5, Fig. 5.)

Let AB be an arc of a cycloid, which a body D, by vibrating in a non-resisting medium, shall describe in any time. Bisection that arc in C, so that C may be the lowest point thereof; and the accelerative force with which the body is urged in any place D, or d, or E, will be as the length of the arc CD, or Cd, or CE. Let that force be expressed by that same arc; and since the resistance is as the moment of the time, and therefore given, let it be expressed by the given part CO of the cycloidal arc, and take the arc Od in the same ratio to the arc CD that the arc OB has to the arc CB: and the force with which the body in d is urged in a resisting medium, being the excess of the force Cd above the resistance OC, will be expressed by the arc Od, and will therefore be to the force with which the body D is urged in a non-resisting medium in the place D, as the arc Od to the arc CD; and therefore also in the place B, as the arc OB to the arc CB. Therefore if two bodies D, d, go from the place B, and are urged by these forces; since the forces at the beginning are as the arcs CB and OB, the first velocities and arcs first described will be in the same ratio. Let those arcs be BD and Bd, and the remaining arcs CD, Od, will be in the same ratio. Therefore the forces, being proportional to those arcs CD, Od, will remain in the same ratio as at the beginning, and therefore the bodies will continue describing together arcs in the same ratio. Therefore the forces and velocities, and the remaining arc CD, Od, will be always as the whole arcs CB, OB, and therefore those remaining arcs will be described

together. Therefore the two bodies D and d will arrive together at the places C and O; that which moves in the non-resisting medium, at the place C, and the other, in the resisting medium, at the place O. Now since the velocities in C and O are as the arcs CB, OB, the arcs which the bodies describe when they go farther will be in the same ratio. Let those arcs be CE and Oe. The force with which the body D in a non-resisting medium is retarded in E is as CE; and the force with which the body d in the resisting medium is retarded in e, is as the sum of the force Ce and the resistance CO, that is, as Oe; and therefore the forces with which the bodies are retarded are as the arcs CB, OB, proportional to the arcs CE, Oe; and therefore the velocities, retarded in that given ratio, remain in the same given ratio. Therefore the velocities and the arcs described with those velocities are always to each other in that given ratio of the arcs CB and OB; and therefore if the entire arcs AB, aB, are taken in the same ratio, the bodies D and d will describe those arcs together, and in the places A and a will lose all their motion together. Therefore the whole oscillations are isochronal, or are performed in equal times; and any parts of the arcs, as BD, Bd, or BE, Be, that are described together, are proportional to the whole arcs BA, Ba. Q.E.D.

COR. Therefore the swiftest motion in a resisting medium does not fall upon the lowest point C, but is found in that point O, in which the whole arc described Ba is bisected. And the body proceeding from thence to a, is retarded at the same rate with which it was accelerated before in its descent from B to O.

PROPOSITION XXVI. THEOREM XXI.

Funependulous bodies, that are resisted in the ratio of the velocity, have their oscillations in a cycloid isochronal.

For if two bodies, equally distant from their centres of suspension, describe, in oscillating, unequal arcs, and the velocities in the correspondent parts of the arcs be to each other as the whole arcs; the resistances, proportional to the velocities, will be also to each other as the same arcs. Therefore if these resistances be subducted from or added to the motive

forces arising from gravity which are as the same arcs, the differences or sums will be to each other in the same ratio of the arcs; and since the increments and decrements of the velocities are as these differences or sums, the velocities will be always as the whole arcs; therefore if the velocities are in any one case as the whole arcs, they will remain always in the same ratio. But at the beginning of the motion, when the bodies begin to descend and describe those arcs, the forces, which at that time are proportional to the arcs, will generate velocities proportional to the arcs. Therefore the velocities will be always as the whole arcs to be described, and therefore those arcs will be described in the same time. Q.E.D.

PROPOSITION XXVII. THEOREM XXII.

If funependulous bodies are resisted in the duplicate ratio of their velocities, the differences between the times of the oscillations in a resisting medium, and the times of the oscillations in a non-resisting medium of the same specific gravity, will be proportional to the arcs described in oscillating nearly.

For let equal pendulums in a resisting medium describe the unequal arcs A, B; and the resistance of the body in the arc A will be to the resistance of the body in the correspondent part of the arc B in the duplicate ratio of the velocities, that is, as AA to BB nearly. If the resistance in the arc B were to the resistance in the arc A as AB to AA, the times in the arcs A and B would be equal (by the last prop.) Therefore the resistance AA in the arc A, or AB in the arc B, causes the excess of the time in the arc A above the time in a non-resisting medium; and the resistance BB causes the excess of the time in the arc B above the time in a non-resisting medium. But those excesses are as the efficient forces AB and BB nearly, that is, as the arcs A and B. Q.E.D.

COR. 1. Hence from the times of the oscillations in unequal arcs in a resisting medium, may be known the times of the oscillations in a non-resisting medium of the same specific gravity. For the difference of the times will be to the excess of the time in the lesser arc above the time in a non-resisting medium, as the difference of the arcs to the lesser arc.

COR. 2. The shorter oscillations are more isochronal, and very short ones are performed nearly in the same times as in a non-resisting medium. But the times of those which are performed in greater arcs are a little greater, because the resistance in the descent of the body, by which the time is prolonged, is greater, in proportion to the length described in the descent than the resistance in the subsequent ascent, by which the time is contracted. But the time of the oscillations, both short and long, seems to be prolonged in some measure by the motion of the medium. For retarded bodies are resisted somewhat less in proportion to the velocity, and accelerated bodies somewhat more than those that proceed uniformly forwards; because the medium, by the motion it has received from the bodies, going forwards the same way with them, is more agitated in the former case, and less in the latter; and so conspires more or less with the bodies moved. Therefore it resists the pendulums in their descent more, and in their ascent less, than in proportion to the velocity; and these two causes concurring prolong the time.

PROPOSITION XXVIII. THEOREM XXIII.

If a funependulous body, oscillating in a cycloid, be resisted in the ratio of the moments of the time, its resistance will be to the force of gravity as the excess of the arc described in the whole descent above the arc described in the subsequent ascent to twice the length of the pendulum. (Pl. 5, Fig. 5.)

Let BC represent the arc described in the descent, Ca the arc described in the ascent, and Aa the difference of the arcs: and things remaining as they were constructed and demonstrated in prop. 25, the force with which the oscillating body is urged in any place D will be to the force of resistance as the arc CD to the arc CO, which is half of that difference Aa. Therefore the force with which the oscillating body is urged at the beginning or the highest point of the cycloid, that is, the force of gravity, will be to the resistance as the arc of the cycloid, between that highest point and lowest point C, is to the arc CO; that is (doubling those arcs), as the whole cycloidal arc, or twice the length of the pendulum, to the arc Aa. Q.E.D.

PROPOSITION XXIX. PROBLEM VI.

Supposing that a body oscillating in a cycloid is resisted in a duplicate ratio of the velocity : to find the resistance in each place. (Pl. 5, Fig. 6.)

Let Ba be an arc described in one entire oscillation, C the lowest point of the cycloid, and CZ half the whole cycloidal arc, equal to the length of the pendulum ; and let it be required to find the resistance of the body in any place D. Cut the indefinite right line OQ in the points O, S, P, Q, so that (erecting the perpendiculars OK, ST, PI, QE, and with the centre O, and the asymptotes OK, OQ, describing the hyperbola TIGE cutting the perpendiculars ST, PI, QE in T, I, and E, and through the point I drawing KF, parallel to the asymptote OQ, meeting the asymptote OK in K, and the perpendiculars ST and QE in L and F) the hyperbolic area PIEQ may be to the hyperbolic area PITS as the arc BC, described in the descent of the body, to the arc Ca described in the ascent ; and that the area IEF may be to the area ILT as OQ to OS. Then with the perpendicular MN cut off the hyperbolic area PINM, and let that area be to the hyperbolic area PIEQ as the arc CZ to the arc BC described in the descent. And if the perpendicular RG cut off the hyperbolic area PIGR, which shall be to the area PIEQ as any arc CD to the arc BC described in the whole descent, the resistance in any place D will be to the force of gravity as the area $\frac{OR}{OQ}$ IEF — IGH to the area PINM.

For since the forces arising from gravity with which the body is urged in the places Z, B, D, a, are as the arcs CZ, CB, CD, Ca, and those arcs are as the areas PINM, PIEQ, PIGR, PITS ; let those areas be the exponents both of the arcs and of the forces respectively. Let Dd be a very small space described by the body in its descent ; and let it be expressed by the very small area RGgr comprehended between the parallels RG, rg ; and produce rg to h, so that GHhg and RGgr may be the contemporaneous decrements of the areas IGH, PIGR. And the increment GHhg — $\frac{Rr}{OQ}$ IEF,

or $Rr \times HG - \frac{Rr}{OQ} IEF$, of the area $\frac{OR}{OQ} IEF - IGH$, will be to the decrement $RGgr$, or $Rr \times RG$, of the area $PIGR$, as $HG - \frac{IEF}{OQ}$ to RG ; and, therefore, as $OR \times HG - \frac{OR}{OQ} IEF$ to $OR \times GR$ or $OP \times PI$, that is (because of the equal quantities $OR \times HG$, $OR \times HR - OR \times GR$, $ORHK - OPIK$, $PIHR$ and $PIGR + IGH$) as $PIGR + IGH - \frac{OR}{OQ} IEF$ to $OPIK$. Therefore if the area $\frac{OR}{OQ} IEF - IGH$ be called Y , and $RGgr$ the decrement of the area $PIGR$ be given, the increment of the area Y will be as $PIGR - Y$.

Then, if V represent the force arising from the gravity, proportional to the arc CD to be described, by which the body is acted upon in D , and R be put for the resistance, $V - R$ will be the whole force with which the body is urged in D . Therefore the increment of the velocity is as $V - R$ and the particle of time in which it is generated conjunctly. But the velocity itself is as the contemporaneous increment of the space described directly and the same particle of time inversely. Therefore, since the resistance is, by the supposition, as the square of the velocity, the increment of the resistance will (by lem. 2) be as the velocity and the increment of the velocity conjunctly, that is, as the moment of the space and $V - R$ conjunctly; and, therefore, if the moment of the space be given, as $V - R$; that is, if for the force V we put its exponent $PIGR$, and the resistance R be expressed by any other area Z , as $PIGR - Z$.

Therefore the area $PIGR$ uniformly decreasing by the subtraction of given moments, the area Y increases in proportion of $PIGR - Y$, and the area Z in proportion of $PIGR - Z$. And therefore if the areas Y and Z begin together, and at the beginning are equal, these, by the addition of equal moments, will continue to be equal; and in like manner decreasing by equal moments, will vanish together. And, *vice versa*, if they together begin and vanish, they will have equal mo-

ments and be always equal ; and that, because if the resistance Z be augmented, the velocity together with the arc Ca , described in the ascent of the body, will be diminished ; and the point in which all the motion, together with the resistance, ceases coming nearer to the point C , the resistance vanishes sooner than the area Y . And the contrary will happen when the resistance is diminished.

Now the area Z begins and ends where the resistance is nothing, that is, at the beginning of the motion where the arc CD is equal to the arc CB , and the right line RG falls upon the right line QE ; and at the end of the motion where the arc CD is equal to the arc Ca , and RG falls upon the right line ST . And the area Y or $\frac{OR}{OQ}$ $IEF - IGH$ begins and ends also where the resistance is nothing, and therefore where $\frac{OR}{OQ}$ IEF and IGH are equal ; that is (by the construction), where the right line RG falls successively upon the right lines QE and ST . Therefore those areas begin and vanish together, and are therefore always equal. Therefore the area $\frac{OR}{OQ}$ $IEF - IGH$ is equal to the area Z , by which the resistance is expressed, and therefore is to the area $PINM$, by which the gravity is expressed, as the resistance to the gravity. *Q.E.D.*

COR. 1. Therefore the resistance in the lowest place C is to the force of gravity as the area $\frac{OP}{OQ}$ IEF to the area $PINM$.

COR. 2. But it becomes greatest where the area $PIHR$ is to the area IEF as OR to OQ . For in that case its moment (that is, $PIGR - Y$) becomes nothing.

COR. 3. Hence also may be known the velocity in each place, as being in the subduplicate ratio of the resistance, and at the beginning of the motion equal to the velocity of the body oscillating in the same cycloid without any resistance.

However, by reason of the difficulty of the calculation by which the resistance and the velocity are found by this pro-

position, we have thought fit to subjoin the proposition following.

PROPOSITION XXX. THEOREM XXIV.

If a right line aB (Pl. 6, Fig. 1) be equal to the arc of a cycloid which an oscillating body describes, and at each of its points D the perpendiculars DK be erected, which shall be to the length of the pendulum as the resistance of the body in the corresponding points of the arc to the force of gravity; I say, that the difference between the arc described in the whole descent and the arc described in the whole subsequent ascent drawn into half the sum of the same arcs, will be equal to the area BKa which all those perpendiculars take up.

Let the arc of the cycloid, described in one entire oscillation, be expressed by the right line aB, equal to it, and the arc which would have been described *in vacuo* by the length AB. Bisect AB in C, and the point C will represent the lowest point of the cycloid, and CD will be as the force arising from gravity, with which the body in D is urged in the direction of the tangent of the cycloid, and will have the same ratio to the length of the pendulum as the force in D has to the force of gravity. Let that force, therefore, be expressed by that length CD, and the force of gravity by the length of the pendulum; and if in DE you take DK in the same ratio to the length of the pendulum as the resistance has to the gravity, DK will be the exponent of the resistance. From the centre C with the interval CA or CB describe a semicircle BEcA. Let the body describe, in the least time, the space Dd; and, erecting the perpendiculars DE, de, meeting the circumference in E and e, they will be as the velocities which the body descending *in vacuo* from the point B would acquire in the places D and d. This appears by prop. 52, book 1. Let, therefore, these velocities be expressed by those perpendiculars DE, de; and let DF be the velocity which it acquires in D by falling from B in the resisting medium. And if from the centre C, with the interval CF, we describe the circle FfM, meeting the right lines de and AB in f and M, then M will be the place to which it would thenceforward, without far-

ther resistance, ascend, and *df* the velocity it would acquire in *d*. Whence, also, if *Fg* represent the moment of the velocity which the body *D*, in describing the least space *Dd*, loses by the resistance of the medium; and *CN* be taken equal to *Cg*; then will *N* be the place to which the body, if it met no farther resistance, would thenceforward ascend, and *MN* will be the decrement of the ascent arising from the loss of that velocity. Draw *Fm* perpendicular to *df*, and the decrement *Fg* of the velocity *DF* generated by the resistance *DK* will be to the increment *fm* of the same velocity, generated by the force *CD*, as the generating force *DK* to the generating force *CD*. But because of the similar triangles *Fmf*, *Fhg*, *FDC*, *fm* is to *Fm* or *Dd* as *CD* to *DF*; and, *ex æquo*, *Fg* to *Dd* as *DK* to *DF*. Also *Fh* is to *Fg* as *DF* to *CF*; and, *ex æquo perturbatè*, *Fh* or *MN* to *Dd* as *DK* to *CF* or *CM*; and therefore the sum of all the *MN* $\times$ *CM* will be equal to the sum of all the *Dd* $\times$ *DK*. At the moveable point *M* suppose always a rectangular ordinate erected equal to the indeterminate *CM*, which, by a continual motion, is drawn into the whole length *Aa*; and the trapezium described by that motion, or its equal, the rectangle *Aa* $\times$ $\frac{1}{2}aB$, will be equal to the sum of all the *MN* $\times$ *CM*, and therefore to the sum of all the *Dd* $\times$ *DK*, that is, to the area *BKVTa*. Q.E.D.

COR. Hence, from the law of resistance, and the difference *Aa* of the arcs *Ca*, *CB*, may be collected the proportion of the resistance to the gravity nearly.

For, if the resistance *DK* be uniform, the figure *BKVTa* will be a rectangle under *Ba* and *DK*; and thence the rectangle under $\frac{1}{2}Ba$ and *Aa* will be equal to the rectangle under *Ba* and *DK*, and *DK* will be equal to $\frac{1}{2}Aa$. Wherefore since *DK* is the exponent of the resistance, and the length of the pendulum the exponent of the gravity, the resistance will be to the gravity as $\frac{1}{2}Aa$ to the length of the pendulum; altogether as in prop. 28 is demonstrated.

If the resistance be as the velocity, the figure *BKVTa* will be nearly an ellipsis. For, if a body, in a non-resisting medium, by one entire oscillation, should describe the length *BA*, the velocity in any place *D* would be as the ordinate

DE of the circle described on the diameter AB. Therefore since Ba in the resisting medium, and BA in the non-resisting one, are described nearly in the same times; and therefore the velocities in each of the points of Ra are to the velocities in the correspondent points of the length BA nearly as Ba is to BA; the velocity in the point D in the resisting medium will be as the ordinate of the circle or ellipsis described upon the diameter Ba; and therefore the figure BKVTa will be nearly an ellipsis. Since the resistance is supposed proportional to the velocity, let OV be the exponent of the resistance in the middle point O; and an ellipsis BRV Sa described with the centre O, and the semi-axes OB, OV, will be nearly equal to the figure BKVTa, and to its equal the rectangle Aa $\times$ BO. Therefore Aa $\times$ BO is to OV $\times$ BO as the area of this ellipsis to OV $\times$ BO; that is, Aa is to OV as the area of the semicircle to the square of the radius, or as 11 to 7 nearly; and therefore $\frac{7}{11}$ Aa is to the length of the pendulum as the resistance of the oscillating body in O to its gravity.

Now, if the resistance DK be in the duplicate ratio of the velocity, the figure BKVTa will be almost a parabola having V for its vertex and OV for its axis, and therefore will be nearly equal to the rectangle under $\frac{2}{3}$ Ba and OV. Therefore the rectangle under $\frac{1}{2}$ Ba and Aa is equal to the rectangle $\frac{2}{3}$ Ba $\times$ OV, and therefore OV is equal to $\frac{3}{4}$ Aa; and therefore the resistance in O made to the oscillating body is to its gravity as $\frac{3}{4}$ Aa to the length of the pendulum.

And I take these conclusions to be accurate enough for practical uses. For, since an ellipsis or parabola BRV Sa falls in with the figure BKVTa in the middle point V, that figure, if greater towards the part BRV or V Sa than the other, is less towards the contrary part, and is therefore nearly equal to it.

PROPOSITION XXXI. THEOREM XXV.

If the resistance made to an oscillating body in each of the proportional parts of the arcs described be augmented or diminished in a given ratio, the difference between the arc described in the descent and the arc described in the subsequent ascent will be augmented or diminished in the same ratio.

For that difference arises from the retardation of the pendulum by the resistance of the medium, and therefore is as the whole retardation and the retarding resistance proportional thereto. In the foregoing proposition the rectangle under the right line $\frac{1}{2}aB$ and the difference Aa of the arcs CB , Ca , was equal to the area $BKTa$. And that area, if the length aB remains, is augmented or diminished in the ratio of the ordinates DK ; that is, in the ratio of the resistance, and is, therefore, as the length aB and the resistance conjunctly. And therefore the rectangle under Aa and $\frac{1}{2}aB$ is as aB and the resistance conjunctly, and therefore Aa is as the resistance. Q.E.D.

COR. 1. Hence, if the resistance be as the velocity, the difference of the arcs in the same medium will be as the whole arc described; and the contrary.

COR. 2. If the resistance be in the duplicate ratio of the velocity, that difference will be in the duplicate ratio of the whole arc; and the contrary.

COR. 3. And universally, if the resistance be in the triplicate or any other ratio of the velocity, the difference will be in the same ratio of the whole arc; and the contrary.

COR. 4. If the resistance be partly in the simple ratio of the velocity, and partly in the duplicate ratio of the same, the difference will be partly in the ratio of the whole arc, and partly in the duplicate ratio of it; and the contrary. So that the law and ratio of the resistance will be the same for the velocity as the law and ratio of that difference for the length of the arc.

COR. 5. And therefore, if a pendulum describe successively unequal arcs, and we can find the ratio of the increment or decrement of this difference for the length of the arc described, there will be had also the ratio of the increment or decrement of the resistance for a greater or less velocity.

GENERAL SCHOLIUM.

From these propositions we may find the resistance of mediums by pendulums oscillating therein. I found the resistance of the air by the following experiments. I suspended a wooden globe or ball weighing $57\frac{7}{8}$ ounces troy, its diameter $6\frac{1}{2}$ London inches, by a fine thread on a firm hook, so that

the distance between the hook and the centre of oscillation of the globe was $10\frac{1}{2}$ feet. I marked on the thread a point 10 feet and 1 inch distant from the centre of suspension; and even with that point I placed a ruler divided into inches, by the help whereof I observed the lengths of the arcs described by the pendulum. Then I numbered the oscillations in which the globe would lose $\frac{1}{8}$ part of its motion. If the pendulum was drawn aside from the perpendicular to the distance of 2 inches, and thence let go, so that in its whole descent it described an arc of 2 inches, and in the first whole oscillation, compounded of the descent and subsequent ascent, an arc of almost 4 inches, the same in 164 oscillations lost $\frac{1}{8}$ part of its motion, so as in its last ascent to describe an arc of $1\frac{3}{4}$ inch. If in the first descent it described an arc of 4 inches, it lost $\frac{1}{8}$ part of its motion in 121 oscillations, so as in its last ascent to describe an arc of $3\frac{1}{2}$ inches. If in the first descent it described an arc of 8, 16, 32, or 64 inches, it lost $\frac{1}{8}$ part of its motion in 69, $35\frac{1}{2}$, $18\frac{1}{2}$, $9\frac{3}{8}$ oscillations respectively. Therefore the difference between the arcs described in the first descent and the last ascent was in the 1st, 2d, 3d, 4th, 5th, 6th cases, $\frac{1}{4}$, $\frac{1}{2}$, 1, 2, 4, 8 inches respectively. Divide those differences by the number of oscillations in each case, and in one mean oscillation, wherein an arc of $3\frac{3}{4}$, $7\frac{1}{2}$, 15, 30, 60, 120 inches was described, the difference of the arcs described in the descent and subsequent ascent will be $\frac{1}{856}$, $\frac{1}{428}$, $\frac{1}{214}$, $\frac{1}{107}$, $\frac{1}{53}$, $\frac{1}{26}$ parts of an inch respectively. But these differences in the greater oscillations are in the duplicate ratio of the arcs described nearly, but in lesser oscillations something greater than in that ratio; and therefore (by cor. 2, prop. 31 of this book) the resistance of the globe, when it moves very swift, is in the duplicate ratio of the velocity, nearly; and when it moves slowly, somewhat greater than in that ratio.

Now, let V represent the greatest velocity in any oscillation, and let A, B, and C be given quantities, and let us suppose the difference of the arcs to be $AV + BV^{\frac{3}{2}} + CV^2$. Since the greatest velocities are in the cycloid as $\frac{1}{2}$ the arcs described in oscillating, and in the circle as $\frac{1}{2}$ the chords of those arcs; and

therefore in equal arcs are greater in the cycloid than in the circle in the ratio of $\frac{1}{2}$ the arcs to their chords; but the times in the circle are greater than in the cycloid, in a reciprocal ratio of the velocity; it is plain that the differences of the arcs (which are as the resistance and the square of the time conjunctly) are nearly the same in both curves: for in the cycloid those differences must be on the one hand augmented, with the resistance, in about the duplicate ratio of the arc to the chord, because of the velocity augmented in the simple ratio of the same; and on the other hand diminished, with the square of the time, in the same duplicate ratio. Therefore to reduce these observations to the cycloid, we must take the same differences of the arcs as were observed in the circle, and suppose the greatest velocities analogous to the half, or the whole arcs, that is, to the numbers $\frac{1}{2}$, 1, 2, 4, 8, 16. Therefore in the 2d, 4th, and 6th cases, put 1, 4, and 16 for V; and the difference of the arcs in the 2d case will become $\frac{\frac{1}{2}}{121} = A + B + C$; in the 4th case, $\frac{2}{35\frac{1}{2}} = 4A + 8B + 16C$; in the 6th case, $\frac{8}{9\frac{2}{3}} = 16A + 64B + 256C$. These equations reduced give $A = 0,0000916$, $B = 0,0010847$, and $C = 0,0029558$. Therefore the difference of the arcs is as $0,0000916V + 0,0010847V^{\frac{3}{2}} + 0,0029558V^2$: and therefore since (by cor. prop. 30, applied to this case) the resistance of the globe in the middle of the arc described in oscillating, where the velocity is V, is to its weight as $\frac{1}{11}AV + \frac{7}{10}BV^{\frac{3}{2}} + \frac{3}{4}CV^2$ to the length of the pendulum, if for A, B, and C you put the numbers found, the resistance of the globe will be to its weight as $0,0000583V + 0,0007593V^{\frac{3}{2}} + 0,0022169V^2$ to the length of the pendulum between the centre of suspension and the ruler, that is, to 121 inches. Therefore, since V in the 2d case represents 1, in the 4th case 4, and in the 6th case 16, the resistance will be to the weight of the globe, in the 2d case, as 0,0030345 to 121; in the 4th, as 0,041748 to 121; in the 6th, as 0,61705 to 121.

The arc which the point marked in the thread described in the 6th case, was of $120 - \frac{8}{9\frac{2}{3}}$, or $119\frac{5}{27}$ inches. And

therefore since the radius was 121 inches, and the length of the pendulum between the point of suspension and the centre of the globe was 126 inches, the arc which the centre of the globe described was $124\frac{3}{4}$ inches. Because the greatest velocity of the oscillating body, by reason of the resistance of the air, does not fall on the lowest point of the arc described, but near the middle place of the whole arc, this velocity will be nearly the same as if the globe in its whole descent in a non-resisting medium should describe $62\frac{3}{8}$ inches, the half of that arc, and that in a cycloid, to which we have above reduced the motion of the pendulum; and therefore that velocity will be equal to that which the globe would acquire by falling perpendicularly from a height equal to the versed sine of that arc. But that versed sine in the cycloid is to that arc $62\frac{3}{8}$ as the same arc to twice the length of the pendulum 252, and therefore equal to 15,278 inches. Therefore the velocity of the pendulum is the same which a body would acquire by falling, and in its fall describing a space of 15,278 inches. Therefore with such a velocity the globe meets with a resistance which is to its weight as 0,61705 to 121, or (if we take that part only of the resistance which is in the duplicate ratio of the velocity) as 0,56752 to 121.

I found, by an hydrostatical experiment, that the weight of this wooden globe was to the weight of a globe of water of the same magnitude as 55 to 97: and therefore since 121 is to 213,4 in the same ratio, the resistance made to this globe of water, moving forwards with the abovementioned velocity, will be to its weight as 0,56752 to 213,4, that is, as 1 to $376\frac{1}{5}$. Whence since the weight of a globe of water, in the time in which the globe with a velocity uniformly continued describes a length of 30,556 inches, will generate all that velocity in the falling globe, it is manifest that the force of resistance uniformly continued in the same time will take away a velocity, which will be less than the other in the ratio of 1 to $376\frac{1}{5}$,

that is, the $\frac{1}{376\frac{1}{50}}$ part of the whole velocity. And therefore in the time that the globe, with the same velocity uniformly continued, would describe the length of its semidiameter, or $3\frac{7}{16}$ inches, it would lose the $\frac{1}{33\frac{1}{2}}$ part of its motion.

I also counted the oscillations in which the pendulum lost $\frac{1}{4}$ part of its motion. In the following table the upper numbers denote the length of the arc described in the first descent, expressed in inches and parts of an inch; the middle numbers denote the length of the arc described in the last ascent; and in the lowest place are the numbers of the oscillations. I give an account of this experiment, as being more accurate than that in which only $\frac{1}{8}$ part of the motion was lost. I leave the calculation to such as are disposed to make it.

<i>First descent</i>	2	4	8	16	32	64
<i>Last ascent</i>	$1\frac{1}{2}$	3	6	12	24	48
<i>Numb. of oscill.</i>	374	272	$162\frac{1}{2}$	$83\frac{1}{3}$	$41\frac{2}{3}$	$22\frac{1}{2}$

I afterwards suspended a leaden globe of 2 inches in diameter, weighing $26\frac{1}{4}$ ounces troy by the same thread, so that between the centre of the globe and the point of suspension there was an interval of $10\frac{1}{2}$ feet, and I counted the oscillations in which a given part of the motion was lost. The first of the following tables exhibits the number of oscillations in which $\frac{1}{8}$ part of the whole motion was lost; the second the number of oscillations in which there was lost $\frac{1}{4}$ part of the same.

<i>First descent</i>	1	2	4	8	16	32	64
<i>Last ascent</i>	$\frac{7}{8}$	$\frac{7}{4}$	$3\frac{1}{2}$	7	14	28	56
<i>Numb. of oscill.</i>	226	228	193	140	$90\frac{1}{2}$	53	30
<i>First descent</i>	1	2	4	8	16	32	64
<i>Last ascent</i>	$\frac{3}{4}$	$1\frac{1}{2}$	3	6	12	24	48
<i>Numb. of oscill.</i>	510	518	420	318	204	121	70

Selecting in the first table the 3d, 5th, and 7th observations, and expressing the greatest velocities in these observations particularly by the numbers 1, 4, 16 respectively, and generally by the quantity V as above, there will come out in the 3d observation $\frac{V^{\frac{1}{2}}}{193} = A + B + C$, in the 5th observation $\frac{2}{90\frac{1}{2}} =$

$4A + 8B + 16C$, in the 7th observation $\frac{8}{30} = 16A + 64B + 256C$. These equations reduced give $A = 0,001414$, $B = 0,000297$, $C = 0,000879$. And thence the resistance of the globe moving with the velocity V will be to its weight $26\frac{1}{4}$ ounces in the same ratio as $0,0009V + 0,000208V^{\frac{3}{2}} + 0,000659V^2$ to 121 inches, the length of the pendulum. And if we regard that part only of the resistance which is in the duplicate ratio of the velocity, it will be to the weight of the globe as $0,000659V^2$ to 121 inches. But this part of the resistance in the 1st experiment was to the weight of the wooden globe of $57\frac{7}{22}$ ounces as $0,002217V^2$ to 121; and thence the resistance of the wooden globe is to the resistance of the leaden one (their velocities being equal) as $57\frac{7}{22}$ into $0,002217$ to $26\frac{1}{4}$ into $0,000659$, that is, as $7\frac{1}{3}$ to 1. The diameters of the two globes were $6\frac{1}{8}$ and 2 inches, and the squares of these are to each other as $47\frac{1}{4}$ and 4, or $11\frac{13}{16}$ and 1, nearly. Therefore the resistances of these equally swift globes were in less than a duplicate ratio of the diameters. But we have not yet considered the resistance of the thread, which was certainly very considerable, and ought to be subducted from the resistance of the pendulums here found. I could not determine this accurately; but I found it greater than a third part of the whole resistance of the lesser pendulum; and thence I gathered that the resistances of the globes, when the resistance of the thread is subducted, are nearly in the duplicate ratio of their diameters. For the ratio of $7\frac{1}{3} - \frac{1}{3}$ to $1 - \frac{1}{3}$, or $10\frac{1}{2}$ to 1, is not very different from the duplicate ratio of the diameters $11\frac{13}{16}$ to 1.

Since the resistance of the thread is of less moment in greater globes, I tried the experiment also with a globe whose diameter was $18\frac{1}{4}$ inches. The length of the pendulum between the point of suspension and the centre of oscillation was $122\frac{1}{2}$ inches, and between the point of suspension and the knot in the thread $109\frac{1}{2}$ inches. The arc described by the knot at the first descent of the pendulum was 32 inches. The arc described by the same knot in the last ascent after five of-

cillations was 28 inches. The sum of the arcs, or the whole arc described in one mean oscillation, was 60 inches. The difference of the arcs 4 inches. The $\frac{1}{10}$ part of this, or the difference between the descent and ascent in one mean oscillation, is $\frac{2}{5}$ of an inch. Then as the radius $109\frac{1}{2}$ to the radius $122\frac{1}{2}$, so is the whole arc of 60 inches described by the knot in one mean oscillation to the whole arc of $67\frac{1}{8}$ inches described by the centre of the globe in one mean oscillation; and so is the difference $\frac{2}{5}$ to a new difference 0,4475. If the length of the arc described were to remain, and the length of the pendulum should be augmented in the ratio of 126 to $122\frac{1}{2}$, the time of the oscillation would be augmented, and the velocity of the pendulum would be diminished in the subduplicate of that ratio; so that the difference 0,4475 of the arcs described in the descent and subsequent ascent would remain. Then if the arc described be augmented in the ratio of $124\frac{3}{31}$ to $67\frac{1}{8}$, that difference 0,4475 would be augmented in the duplicate of that ratio, and so would become 1,5295. These things would be so upon the supposition that the resistance of the pendulum were in the duplicate ratio of the velocity. Therefore if the pendulum describe the whole arc of $124\frac{3}{31}$ inches, and its length between the point of suspension and the centre of oscillation be 126 inches, the difference of the arcs described in the descent and subsequent ascent would be 1,5295 inches. And this difference multiplied into the weight of the pendulous globe, which was 208 ounces, produces 318,136. Again; in the pendulum abovementioned, made of a wooden globe, when its centre of oscillation, being 126 inches from the point of suspension, described the whole arc of $124\frac{3}{31}$ inches, the difference of the arcs described in the descent and ascent was $\frac{126}{121}$ into $\frac{8}{9\frac{2}{3}}$. This multiplied into the weight of the globe, which was $57\frac{7}{22}$ ounces, produces 49,396. But I multiply these differences into the weights of the globes, in order to find their resistances. For the differences arise from the resistances, and are as the resistances directly and the weights inversely. Therefore the resistances are as the numbers 318,136 and 49,396. But that part of the resistance of the lesser globe,

which is in the duplicate ratio of the velocity, was to the whole resistance as 0,56752 to 0,61675, that is, as 45,453 to 49,396; whereas that part of the resistance of the greater globe is almost equal to its whole resistance; and so those parts are nearly as 318,136 and 45,453, that is, as 7 and 1. But the diameters of the globes are $18\frac{3}{4}$ and $6\frac{7}{8}$; and their squares $351\frac{9}{16}$ and $47\frac{17}{64}$ are as 7,438 and 1, that is, as the resistances of the globes 7 and 1, nearly. The difference of these ratios is scarce greater than may arise from the resistance of the thread. Therefore those parts of the resistances which are, when the globes are equal, as the squares of the velocities, are also, when the velocities are equal, as the squares of the diameters of the globes.

But the greatest of the globes I used in these experiments was not perfectly spherical, and therefore in this calculation I have, for brevity's sake, neglected some little niceties, being not very solicitous for an accurate calculus in an experiment that was not very accurate; so that I could wish that these experiments were tried again with other globes, of a larger size, more in number, and more accurately formed, since the demonstration of a vacuum depends thereon. If the globes be taken in a geometrical proportion, as suppose whose diameters are 4, 8, 16, 32 inches, one may collect from the progression observed in the experiments what would happen if the globes were still larger.

In order to compare the resistances of different fluids with each other, I made the following trials. I procured a wooden vessel 4 feet long, 1 foot broad, and 1 foot high. This vessel, being uncovered, I filled with spring water, and having immersed pendulums therein, I made them oscillate in the water. And I found that a leaden globe weighing $166\frac{1}{2}$ ounces, and in diameter $3\frac{3}{4}$ inches, moved therein as it is set down in the following table; the length of the pendulum from the point of suspension to a certain point marked in the thread being 126 inches, and to the centre of oscillation $134\frac{3}{8}$ inches.

The arc described in
the first descent by
a point marked in
the thread, was
inches

64 . 32 . 16 . 8 . 4 . 2 . 1 . $\frac{1}{2}$. $\frac{1}{4}$

The arc described in
the last ascent was
inches

48 . 24 . 12 . 6 . 3 . $1\frac{1}{2}$. $\frac{3}{4}$. $\frac{3}{8}$. $\frac{3}{16}$

The difference of the
arcs proportional
to the motion lost,
was inches.....

16 . 8 . 4 . 2 . 1 . $\frac{1}{2}$. $\frac{1}{4}$. $\frac{1}{8}$. $\frac{1}{16}$

The number of the
oscillations in wa-
ter

$\frac{32}{60}$. $1\frac{1}{5}$. 3 . 7 . $11\frac{1}{4}$. $12\frac{2}{3}$. $13\frac{1}{3}$

The number of the of-
cillations in air..

$85\frac{1}{2}$. 287 . 535

In the experiments of the 4th column there were equal motions lost in 535 oscillations made in the air, and $1\frac{1}{5}$ in water. The oscillations in the air were indeed a little swifter than those in the water. But if the oscillations in the water were accelerated in such a ratio that the motions of the pendulums might be equally swift in both mediums, there would be still the same number $1\frac{1}{5}$ of oscillations in the water, and by these the same quantity of motion would be lost as before; because the resistance is increased, and the square of the time diminished in the same duplicate ratio. The pendulums, therefore, being of equal velocities, there were equal motions lost in 535 oscillations in the air, and $1\frac{1}{5}$ in the water: and therefore the resistance of the pendulum in the water is to its resistance in the air as 535 to $1\frac{1}{5}$. This is the proportion of the whole resistances in the case of the 4th column.

Now let $AV + CV^2$ represent the difference of the arcs described in the descent and subsequent ascent by the globe moving in air with the greatest velocity V ; and since the greatest velocity is in the case of the 4th column to the greatest velocity in the case of the 1st column as 1 to 8; and that difference of the arcs in the case of the 4th column to the dif-

ference in the case of the 1st column as $\frac{2}{535}$ to $\frac{16}{85\frac{1}{2}}$, or as $85\frac{1}{2}$ to 4280; put in these cases 1 and 8 for the velocities, and $85\frac{1}{2}$ and 4280 for the difference of the arcs, and $A + C$ will be $= 85\frac{1}{2}$, and $8A + 64C = 4280$ or $A + 8C = 535$; and then, by reducing these equations, there will come out $7C = 449\frac{1}{2}$ and $C = 64\frac{3}{14}$ and $A = 21\frac{2}{7}$; and therefore the resistance, which is as $\frac{7}{11}AV + \frac{3}{4}CV^2$, will become as $13\frac{6}{11}V = 48\frac{9}{56}V^2$. Therefore in the case of the 4th column, where the velocity was 1, the whole resistance is to its part proportional to the square of the velocity as $13\frac{6}{11} + 48\frac{9}{56}$ or $61\frac{12}{7}$ to $48\frac{9}{56}$; and therefore the resistance of the pendulum in water is to that part of the resistance in air, which is proportional to the square of the velocity, and which in swift motions is the only part that deserves consideration, as $61\frac{12}{7}$ to $48\frac{9}{56}$ and 535 to $1\frac{1}{5}$ conjunctly, that is, as 571 to 1. If the whole thread of the pendulum oscillating in the water had been immersed, its resistance would have been still greater; so that the resistance of the pendulum oscillating in the water, that is, that part which is proportional to the square of the velocity, and which only needs to be considered in swift bodies, is to the resistance of the same whole pendulum, oscillating in air with the same velocity, as about 850 to 1, that is, as the density of water to the density of air, nearly.

In this calculation we ought also to have taken in that part of the resistance of the pendulum in the water which was as the square of the velocity; but I found (which will perhaps seem strange) that the resistance in the water was augmented in more than a duplicate ratio of the velocity. In searching after the cause, I thought upon this, that the vessel was too narrow for the magnitude of the pendulous globe, and by its narrowness obstructed the motion of the water as it yielded to the oscillating globe. For when I immersed a pendulous globe, whose diameter was one inch only, the resistance was augmented nearly in a duplicate ratio of the velocity. I tried this by making a pendulum of two globes, of which the lesser and lower oscillated in the water, and the greater and higher was fastened to the thread just above the

water, and, by oscillating in the air, assisted the motion of the pendulum, and continued it longer. The experiments made by this contrivance proved according to the following table.

<i>Arc described in first descent</i> ..	16	. 8	. 4	. 2	. 1	. $\frac{1}{2}$	. $\frac{1}{4}$
<i>Arc described in last ascent</i> ...	12	. 6	. 3	. $1\frac{1}{2}$	. $\frac{3}{4}$	. $\frac{3}{8}$	. $\frac{3}{16}$
<i>Diff. of arcs, proportional to motion lost</i>	4	. 2	. 1	. $\frac{1}{2}$	. $\frac{1}{4}$	. $\frac{1}{8}$	. $\frac{1}{16}$
<i>Number of oscillations</i>							
	$3\frac{3}{8}$	. $6\frac{1}{2}$	. $12\frac{1}{2}$	. $21\frac{1}{3}$	. 34	. 53	. $62\frac{1}{2}$

In comparing the resistances of the mediums with each other, I also caused iron pendulums to oscillate in quicksilver. The length of the iron wire was about 3 feet, and the diameter of the pendulous globe about $\frac{1}{3}$ of an inch. To the wire, just above the quicksilver, there was fixed another leaden globe of a bigness sufficient to continue the motion of the pendulum for some time. Then a vessel, that would hold about 3 pounds of quicksilver, was filled by turns with quicksilver and common water, that, by making the pendulum oscillate successively in these two different fluids, I might find the proportion of their resistances; and the resistance of the quicksilver proved to be to the resistance of water as about 13 or 14 to 1; that is, as the density of quicksilver to the density of water. When I made use of a pendulous globe something bigger, as of one whose diameter was about $\frac{1}{2}$ or $\frac{2}{3}$ of an inch, the resistance of the quicksilver proved to be to the resistance of the water as about 12 or 10 to 1. But the former experiment is more to be relied on, because in the latter the vessel was too narrow in proportion to the magnitude of the immersed globe; for the vessel ought to have been enlarged together with the globe. I intended to have repeated these experiments with larger vessels, and in melted metals, and other liquors both cold and hot; but I had not leisure to try all; and besides, from what is already described, it appears sufficiently that the resistance of bodies moving swiftly is nearly proportional to the densities of the fluids in which they move. I do not say accurately; for more tenacious fluids, of equal density, will undoubtedly resist more than those that are more liquid; as cold oil more than warm, warm oil more than rain-water, and water more than spirit of wine. But in liquors

which are sensibly fluid enough, as in air, in salt and fresh water, in spirit of wine, of turpentine, and salts, in oil cleared of its fæces by distillation and warmed, in oil of vitriol, and in mercury, and melted metals, and any other such like, that are fluid enough to retain for some time the motion impressed upon them by the agitation of the vessel, and which being poured out are easily resolved into drops, I doubt not but the rule already laid down may be accurate enough, especially if the experiments be made with larger pendulous bodies, and more swiftly moved.

Lastly, since it is the opinion of some that there is a certain æthereal medium extremely rare and subtle, which freely pervades the pores of all bodies; and from such a medium, so pervading the pores of bodies, some resistance must needs arise; in order to try whether the resistance, which we experience in bodies in motion, be made upon their outward superficies only, or whether their internal parts meet with any considerable resistance upon their superficies, I thought of the following experiment. I suspended a round deal box by a thread 11 feet long, on a steel hook, by means of a ring of the same metal, so as to make a pendulum of the aforesaid length. The hook had a sharp hollow edge on its upper part, so that the upper arc of the ring pressing on the edge might move the more freely; and the thread was fastened to the lower arc of the ring. The pendulum being thus prepared, I drew it aside from the perpendicular to the distance of about 6 feet, and that in a plane perpendicular to the edge of the hook, left the ring, while the pendulum oscillated, should slide to and fro on the edge of the hook: for the point of suspension, in which the ring touches the hook, ought to remain immovable. I therefore accurately noted the place to which the pendulum was brought, and, letting it go, I marked three other places, to which it returned at the end of the 1st, 2d, and 3d oscillation. This I often repeated, that I might find those places as accurately as possible. Then I filled the box with lead and other heavy metals that were near at hand. But, first, I weighed the box when empty, and that part of the thread that went round it, and half the remaining part, extended

between the hook and the suspended box; for the thread so extended always acts upon the pendulum, when drawn aside from the perpendicular, with half its weight. To this weight I added the weight of the air contained in the box. And this whole weight was about $\frac{1}{78}$ of the weight of the box when filled with the metals. Then because the box when full of the metals, by extending the thread with its weight, increased the length of the pendulum, I shortened the thread so as to make the length of the pendulum, when oscillating, the same as before. Then drawing aside the pendulum to the place first marked, and letting it go, I reckoned about 77 oscillations before the box returned to the second mark, and as many afterwards before it came to the third mark, and as many after that before it came to the fourth mark. From whence I conclude that the whole resistance of the box, when full, had not a greater proportion to the resistance of the box, when empty, than 78 to 77. For if their resistances were equal, the box, when full, by reason of its *vis insita*, which was 78 times greater than the *vis insita* of the same when empty, ought to have continued its oscillating motion so much the longer, and therefore to have returned to those marks at the end of 78 oscillations. But it returned to them at the end of 77 oscillations.

Let, therefore, A represent the resistance of the box upon its external superficies, and B the resistance of the empty box on its internal superficies; and if the resistances to the internal parts of bodies equally swift be as the matter, or the number of particles that are resisted, then 78B will be the resistance made to the internal parts of the box, when full; and therefore the whole resistance $A + B$ of the empty box will be to the whole resistance $A + 78B$ of the full box as 77 to 78; and, by division, $A + B$ to 77B as 77 to 1; and thence $A + B$ to B as 77×77 to 1, and, by division again, A to B as 5928 to 1. Therefore the resistance of the empty box in its internal parts will be above 5000 times less than the resistance on its external superficies. This reasoning depends upon the supposition that the greater resistance of the full box arises

not from any other latent cause, but only from the action of some subtile fluid upon the included metal.

This experiment is related by memory, the paper being lost in which I had described it; so that I have been obliged to omit some fractional parts, which are slipped out of my memory; and I have no leisure to try it again. The first time I made it, the hook being weak, the full box was retarded sooner. The cause I found to be, that the hook was not strong enough to bear the weight of the box; so that, as it oscillated to and fro, the hook was bent sometimes this and sometimes that way. I therefore procured a hook of sufficient strength, so that the point of suspension might remain unmoved, and then all things happened as is above described.

SECTION VII.

Of the motion of fluids, and the resistance made to projected bodies.

PROPOSITION XXXII. THEOREM XXVI.

Suppose two similar systems of bodies consisting of an equal number of particles, and let the correspondent particles be similar and proportional, each in one system to each in the other, and have a like situation among themselves, and the same given ratio of density to each other; and let them begin to move among themselves in proportional times, and with like motions (that is, those in one system among one another, and those in the other among one another). And if the particles that are in the same system do not touch one another, except in the moments of reflexion; nor attract nor repel each other, except with accelerative forces that are as the diameters of the correspondent particles inversely, and the squares of the velocities directly; I say, that the particles of those systems will continue to move among themselves with like motions and in proportional times.

Like bodies in like situations are said to be moved among themselves with like motions and in proportional times, when their situations at the end of those times are always found alike in respect of each other; as suppose we compare the particles in one system with the correspondent particles in the other. Hence the times will be proportional, in which simi-

lar and proportional parts of similar figures will be described by correspondent particles. Therefore if we suppose two systems of this kind, the correspondent particles, by reason of the similitude of the motions at their beginning, will continue to be moved with like motions, so long as they move without meeting one another; for if they are acted on by no forces, they will go on uniformly in right lines, by the 1st law. But if they do agitate one another with some certain forces, and those forces are as the diameters of the correspondent particles inversely and the squares of the velocities directly, then, because the particles are in like situations, and their forces are proportional, the whole forces with which correspondent particles are agitated, and which are compounded of each of the agitating forces (by corol. 2 of the laws), will have like directions, and have the same effect as if they respected centres placed alike among the particles; and those whole forces will be to each other as the several forces which compose them, that is, as the diameters of the correspondent particles inversely, and the squares of the velocities directly: and therefore will cause correspondent particles to continue to describe like figures. These things will be so (by cor. 1 and 8, prop. 4, book 1), if those centres are at rest; but if they are moved, yet, by reason of the similitude of the translations, their situations among the particles of the system will remain similar; so that the changes introduced into the figures described by the particles will still be similar. So that the motions of correspondent and similar particles will continue similar till their first meeting with each other; and thence will arise similar collisions, and similar reflexions; which will again beget similar motions of the particles among themselves (by what was just now shewn), till they mutually fall upon one another again, and so on *ad infinitum*.

COR. 1. Hence if any two bodies, which are similar and in like situations to the correspondent particles of the systems, begin to move amongst them in like manner and in proportional times, and their magnitudes and densities be to each other as the magnitudes and densities of the corresponding particles, these bodies will continue to be moved in like man-

ner and in proportional times; for the case of the greater parts of both systems and of the particles is the very same.

COR. 2. And if all the similar and similarly situated parts of both systems be at rest among themselves; and two of them, which are greater than the rest, and mutually correspondent in both systems, begin to move in lines alike posited, with any similar motion whatsoever, they will excite similar motions in the rest of the parts of the systems, and will continue to move among those parts in like manner and in proportional times; and will therefore describe spaces proportional to their diameters.

PROPOSITION XXXIII. THEOREM XXVII.

The same things being supposed, I say, that the greater parts of the systems are resisted in a ratio compounded of the duplicate ratio of their velocities, and the duplicate ratio of their diameters, and the simple ratio of the density of the parts of the systems.

For the resistance arises partly from the centripetal or centrifugal forces with which the particles of the system mutually act on each other, partly from the collisions and reflexions of the particles and the greater parts. The resistances of the first kind are to each other as the whole motive forces from which they arise, that is, as the whole accelerative forces and the quantities of matter in corresponding parts; that is (by the supposition), as the squares of the velocities directly, and the distances of the corresponding particles inversely, and the quantities of matter in the correspondent parts directly: and therefore since the distances of the particles in one system are to the correspondent distances of the particles of the other as the diameter of one particle or part in the former system to the diameter of the correspondent particle or part in the other, and since the quantities of matter are as the densities of the parts and the cubes of the diameters; the resistances are to each other as the squares of the velocities and the squares of the diameters and the densities of the parts of the systems. Q.E.D. The resistances of the latter sort are as the number of correspondent reflexions and the forces of those reflexions conjunctly; but the number

of the reflexions are to each other as the velocities of the corresponding parts directly and the spaces between their reflexions inversely. And the forces of the reflexions are as the velocities and the magnitudes and the densities of the corresponding parts conjunctly; that is, as the velocities and the cubes of the diameters and the densities of the parts. And joining all these ratios, the resistances of the corresponding parts are to each other as the squares of the velocities and the squares of the diameters and the densities of the parts conjunctly. Q.E.D.

COR. 1. Therefore if those systems are two elastic fluids, like our air, and their parts are at rest among themselves; and two similar bodies proportional in magnitude and density to the parts of the fluids, and similarly situated among those parts, be any how projected in the direction of lines similarly posited; and the accelerative forces with which the particles of the fluids mutually act upon each other are as the diameters of the bodies projected inversely and the squares of their velocities directly; those bodies will excite similar motions in the fluids in proportional times, and will describe similar spaces and proportional to their diameters.

COR. 2. Therefore in the same fluid a projected body that moves swiftly meets with a resistance that is in the duplicate ratio of its velocity, nearly. For if the forces with which distant particles act mutually upon one another should be augmented in the duplicate ratio of the velocity, the projected body would be resisted in the same duplicate ratio accurately; and therefore in a medium, whose parts when at a distance do not act mutually with any force on one another, the resistance is in the duplicate ratio of the velocity accurately. Let there be, therefore, three mediums A, B, C, consisting of similar and equal parts regularly disposed at equal distances. Let the parts of the mediums A and B recede from each other with forces that are among themselves as T and V; and let the parts of the medium C be entirely destitute of any such forces. And if four equal bodies D, E, F, G, move in these mediums, the two first D and E in the two first A and B, and the other two F and G in the third C; and if

the velocity of the body D be to the velocity of the body E, and the velocity of the body F to the velocity of the body G, in the subduplicate ratio of the force T to the force V; the resistance of the body D to the resistance of the body E, and the resistance of the body F to the resistance of the body G, will be in the duplicate ratio of the velocities; and therefore the resistance of the body D will be to the resistance of the body F as the resistance of the body E to the resistance of the body G. Let the bodies D and F be equally swift, as also the bodies E and G; and, augmenting the velocities of the bodies D and F in any ratio, and diminishing the forces of the particles of the medium B in the duplicate of the same ratio, the medium B will approach to the form and condition of the medium C at pleasure; and therefore the resistances of the equal and equally swift bodies E and G in these mediums will perpetually approach to equality, so that their difference will at last become less than any given. Therefore, since the resistances of the bodies D and F are to each other as the resistances of the bodies E and G, those will also in like manner approach to the ratio of equality. Therefore the bodies D and F, when they move with very great swiftness, meet with resistances very nearly equal; and therefore since the resistance of the body F is in a duplicate ratio of the velocity, the resistance of the body D will be nearly in the same ratio.

COR. 3. The resistance of a body moving very swift in an elastic fluid is almost the same as if the parts of the fluid were destitute of their centrifugal forces, and did not fly from each other; if so be that the elasticity of the fluid arise from the centrifugal forces of the particles, and the velocity be so great as not to allow the particles time enough to act.

COR. 4. Therefore, since the resistances of similar and equally swift bodies, in a medium whose distant parts do not fly from each other, are as the squares of the diameters, the resistances made to bodies moving with very great and equal velocities in an elastic fluid will be as the squares of the diameters, nearly.

COR. 5. And since similar, equal, and equally swift bodies, moving through mediums of the same density, whose parti-

cles do not fly from each other mutually, will strike against an equal quantity of matter in equal times, whether the particles of which the medium consists be more and smaller, or fewer and greater, and therefore impress on that matter an equal quantity of motion, and in return (by the 3d law of motion) suffer an equal re-action from the same, that is, are equally resisted; it is manifest, also, that in elastic fluids of the same density, when the bodies move with extreme swiftness, their resistances are nearly equal, whether the fluids consist of gross parts, or of parts ever so subtile. For the resistance of projectiles moving with exceedingly great celerities is not much diminished by the subtilty of the medium.

COR. 6. All these things are so in fluids whose elastic force takes its rise from the centrifugal forces of the particles. But if that force arise from some other cause, as from the expansion of the particles after the manner of wool, or the boughs of trees, or any other cause, by which the particles are hindered from moving freely among themselves, the resistance, by reason of the lesser fluidity of the medium, will be greater than in the corollaries above.

PROPOSITION XXXIV. THEOREM XXVIII.

If in a rare medium, consisting of equal particles freely disposed at equal distances from each other, a globe and a cylinder described on equal diameters move with equal velocities in the direction of the axis of the cylinder, the resistance of the globe will be but half so great as that of the cylinder.

For since the action of the medium upon the body is the same (by cor. 5 of the laws) whether the body move in a quiescent medium, or whether the particles of the medium impinge with the same velocity upon the quiescent body, let us consider the body as if it were quiescent, and see with what force it would be impelled by the moving medium. Let, therefore, ABKI (Pl. 6, Fig. 2) represent a spherical body described from the centre C with the semi-diameter CA, and let the particles of the medium impinge with a given velocity upon that spherical body in the directions of right lines parallel to AC; and let FB be one of those right lines. In FB take LB equal to the semi-diameter CB, and draw BD touch-

ing the sphere in B. Upon KC and BD let fall the perpendiculars BE, LD; and the force with which a particle of the medium, impinging on the globe obliquely in the direction FB, would strike the globe in B, will be to the force with which the same particle, meeting the cylinder ONGQ described about the globe with the axis ACI, would strike it perpendicularly in b, as LD to LB, or BE to BC. Again; the efficacy of this force to move the globe according to the direction of its incidence FB or AC, is to the efficacy of the same to move the globe according to the direction of its determination, that is, in the direction of the right line BC in which it impels the globe directly, as BE to BC. And, joining these ratios, the efficacy of a particle, falling upon the globe obliquely in the direction of the right line FB, to move the globe in the direction of its incidence, is to the efficacy of the same particle falling in the same line perpendicularly on the cylinder, to move it in the same direction, as BE^2 to BC^2 . Therefore if in bE, which is perpendicular to the circular base of the cylinder NAO, and equal to the radius AC, we take bH equal to $\frac{BE^2}{CB}$, then bH will be to bE as the ef-

fect of the particle upon the globe to the effect of the particle upon the cylinder. And therefore the solid which is formed by all the right lines bH will be to the solid formed by all the right lines bE as the effect of all the particles upon the globe to the effect of all the particles upon the cylinder. But the former of these solids is a paraboloid whose vertex is C, its axis CA, and latus rectum CA, and the latter solid is a cylinder circumscribing the paraboloid; and it is known that a paraboloid is half its circumscribed cylinder. Therefore the whole force of the medium upon the globe is half of the entire force of the same upon the cylinder. And therefore if the particles of the medium are at rest, and the cylinder and globe move with equal velocities, the resistance of the globe will be half the resistance of the cylinder. Q.E.D.

SCHOLIUM.

By the same method other figures may be compared together as to their resistance; and those may be found which

are most apt to continue their motions in resisting mediums. As if upon the circular base CEBH (Pl. 6, Fig. 3) from the centre O, with the radius OC, and the altitude OD, one would construct a frustum CBGF of a cone, which should meet with less resistance than any other frustum constructed with the same base and altitude, and going forwards towards D in the direction of its axis: bisect the altitude OD in Q, and produce OQ to S, so that QS may be equal to QC, and S will be the vertex of the cone whose frustum is sought.

Whence, by the bye, since the angle CSB is always acute, it follows, that, if the solid ADBE (Pl. 6, Fig. 4) be generated by the convolution of an elliptical or oval figure ADBE about its axis AB, and the generating figure be touched by three right lines FG, GH, HI, in the points F, B, and I, so that GH shall be perpendicular to the axis in the point of contact B, and FG, HI may be inclined to GH in the angles FGB, BHI of 135 degrees; the solid arising from the convolution of the figure ADFGHIE about the same axis AB will be less resisted than the former solid; if so be that both move forward in the direction of their axis AB, and that the extremity B of each go foremost. Which proposition I conceive may be of use in the building of ships.

If the figure DNFG be such a curve, that if, from any point thereof, as N, the perpendicular NM be let fall on the axis AB, and from the given point G there be drawn the right line GR parallel to the right line touching the figure in N, and cutting the axis produced in R, MN becomes to GR as GR^2 to $4BR \times GB^2$, the solid described by the revolution of this figure about its axis AB, moving in the before-mentioned rare medium from A towards B, will be less resisted than any other circular solid whatsoever, described of the same length and breadth.

 The demonstration of these curious Theorems being omitted by the author, the analysis thereof, communicated by a friend, is added at the end of this volume.

PROPOSITION XXXV. PROBLEM VII.

If a rare medium consist of very small quiescent particles of equal magnitudes, and freely disposed at equal distances from

one another: to find the resistance of a globe moving uniformly forwards in this medium.

CASE 1. Let a cylinder described with the same diameter and altitude be conceived to go forward with the same velocity in the direction of its axis through the same medium; and let us suppose that the particles of the medium, on which the globe or cylinder falls, fly back with as great a force of reflexion as possible. Then, since the resistance of the globe (by the last proposition) is but half the resistance of the cylinder, and since the globe is to the cylinder as 2 to 3, and since the cylinder by falling perpendicularly on the particles, and reflecting them with the utmost force, communicates to them a velocity double to its own; it follows that the cylinder, in moving forward uniformly half the length of its axis, will communicate a motion to the particles which is to the whole motion of the cylinder as the density of the medium to the density of the cylinder; and that the globe, in the time it describes one length of its diameter in moving uniformly forwards, will communicate the same motion to the particles; and in the time that it describes two thirds of its diameter, will communicate a motion to the particles which is to the whole motion of the globe as the density of the medium to the density of the globe. And therefore the globe meets with a resistance, which is to the force by which its whole motion may be either taken away or generated in the time in which it describes two thirds of its diameter moving uniformly forwards, as the density of the medium to the density of the globe.

CASE 2. Let us suppose that the particles of the medium incident on the globe or cylinder are not reflected; and then the cylinder falling perpendicularly on the particles will communicate its own simple velocity to them, and therefore meets a resistance but half so great as in the former case, and the globe also meets with a resistance but half so great.

CASE 3. Let us suppose the particles of the medium to fly back from the globe with a force which is neither the greatest, nor yet none at all, but with a certain mean force; then the resistance of the globe will be in the same mean ratio between

the resistance in the first case and the resistance in the second. Q.E.I.

COR. 1. Hence if the globe and the particles are infinitely hard, and destitute of all elastic force, and therefore of all force of reflexion, the resistance of the globe will be to the force by which its whole motion may be destroyed or generated, in the time that the globe describes four third parts of its diameter, as the density of the medium to the density of the globe.

COR. 2. The resistance of the globe, *cæteris paribus*, is in the duplicate ratio of the velocity.

COR. 3. The resistance of the globe, *cæteris paribus*, is in the duplicate ratio of the diameter.

COR. 4. The resistance of the globe is, *cæteris paribus*, as the density of the medium.

COR. 5. The resistance of the globe is in a ratio compounded of the duplicate ratio of the velocity, and the duplicate ratio of the diameter, and the ratio of the density of the medium.

COR. 6. The motion of the globe and its resistance may be thus expounded. Let AB (Pl. 7, Fig. 1) be the time in which the globe may, by its resistance uniformly continued, lose its whole motion. Erect AD, BC perpendicular to AB. Let BC be that whole motion, and through the point C, the asymptotes being AD, AB, describe the hyperbola CF. Produce AB to any point E. Erect the perpendicular EF, meeting the hyperbola in F. Complete the parallelogram CBEG, and draw AF meeting BC in H. Then if the globe in any time BE, with its first motion BC uniformly continued, describes, in a non-resisting medium, the space CBEG expounded by the area of the parallelogram, the same in a resisting medium will describe the space CBEF expounded by the area of the hyperbola; and its motion at the end of that time will be expounded by EF, the ordinate of the hyperbola, there being lost of its motion the part FG. And its resistance at the end of the same time will be expounded by the length BH, there being lost of its resistance the part CH. All these things appear by cor. 1 and 3, prop. 5, book 2.

COR. 7. Hence if the globe in the time T by the resistance R uniformly continued lose its whole motion M , the same globe in the time t in a resisting medium, wherein the resistance R decreases in a duplicate ratio of the velocity, will lose out of its motion M the part $\frac{tM}{T \times t}$, the part $\frac{TM}{T \times t}$ remaining; and will describe a space which is to the space described in the same time t , with the uniform motion M , as the logarithm of the number $\frac{T+t}{T}$ multiplied by the number 2,302585092994 is to the number $\frac{t}{T}$, because the hyperbolic area $BCFE$ is to the rectangle $BCGE$ in that proportion.

SCHOLIUM.

I have exhibited in this proposition the resistance and retardation of spherical projectiles in mediums that are not continued, and shewn that this resistance is to the force by which the whole motion of the globe may be destroyed or produced in the time in which the globe can describe two thirds of its diameter, with a velocity uniformly continued, as the density of the medium to the density of the globe, if so be the globe and the particles of the medium be perfectly elastic, and are endued with the utmost force of reflexion; and that this force, where the globe and particles of the medium are infinitely hard and void of any reflecting force, is diminished one half. But in continued mediums, as water, hot oil, and quicksilver, the globe as it passes through them does not immediately strike against all the particles of the fluid that generate the resistance made to it, but presses only the particles that lie next to it, which press the particles beyond, which press other particles, and so on; and in these mediums the resistance is diminished one other half. A globe in these extremely fluid mediums meets with a resistance that is to the force by which its whole motion may be destroyed or generated in the time wherein it can describe, with that motion uniformly continued, eight third parts of its diameter, as the density of the medium to the density of the globe. This I shall endeavour to shew in what follows.

PROPOSITION XXXVI. PROBLEM VIII.

To define the motion of water running out of a cylindrical vessel through a hole made at the bottom.

Let ACDB (Pl. 7, Fig. 2) be a cylindrical vessel, AB the mouth of it, CD the bottom parallel to the horizon, EF a circular hole in the middle of the bottom, G the centre of the hole, and GH the axis of the cylinder perpendicular to the horizon. And suppose a cylinder of ice APQB to be of the same breadth with the cavity of the vessel, and to have the same axis, and to descend perpetually with an uniform motion, and that its parts, as soon as they touch the superficies AB, dissolve into water, and flow down by their weight into the vessel, and in their fall compose the cataract or column of water ABNFEM, passing through the hole EF, and filling up the same exactly. Let the uniform velocity of the descending ice and of the contiguous water in the circle AB be that which the water would acquire by falling through the space IH; and let IH and HG lie in the same right line; and through the point I let there be drawn the right line KL parallel to the horizon, and meeting the ice on both the sides thereof in K and L. Then the velocity of the water running out at the hole EF will be the same that it would acquire by falling from I through the space IG. Therefore, by *Galileo's* theorems, IG will be to IH in the duplicate ratio of the velocity of the water that runs out at the hole to the velocity of the water in the circle AB, that is, in the duplicate ratio of the circle AB to the circle EF; those circles being reciprocally as the velocities of the water which in the same time and in equal quantities passes severally through each of them, and completely fills them both. We are now considering the velocity with which the water tends to the plane of the horizon. But the motion parallel to the same, by which the parts of the falling water approach to each other, is not here taken notice of; since it is neither produced by gravity, nor at all changes the motion perpendicular to the horizon which the gravity produces. We suppose, indeed, that the parts of the water cohere a little, that by their cohesion they may in falling approach to each other with motions parallel to the horizon, in

order to form one single cataract, and to prevent their being divided into several: but the motion parallel to the horizon arising from this cohesion does not come under our present consideration.

CASE 1. Conceive now the whole cavity in the vessel, which encompasses the falling water ABNFEM, to be full of ice, so that the water may pass through the ice as through a funnel. Then, if the water pass very near to the ice only, without touching it; or, which is the same thing, if, by reason of the perfect smoothness of the surface of the ice, the water, through touching it, glides over it with the utmost freedom, and without the least resistance; the water will run through the hole EF with the same velocity as before, and the whole weight of the column of water ABNFEM will be all taken up, as before, in forcing out the water, and the bottom of the vessel will sustain the weight of the ice encompassing that column.

Let now the ice in the vessel dissolve into water; yet will the efflux of the water remain, as to its velocity, the same as before. It will not be less, because the ice now dissolved will endeavour to descend; it will not be greater, because the ice, now become water, cannot descend without hindering the descent of other water equal to its own descent. The same force ought always to generate the same velocity in the effluent water.

But the hole at the bottom of the vessel, by reason of the oblique motions of the particles of the effluent water, must be a little greater than before. For now the particles of the water do not all of them pass through the hole perpendicularly, but, flowing down on all parts from the sides of the vessel, and converging towards the hole, pass through it with oblique motions; and in tending downwards meet in a stream whose diameter is a little smaller below the hole than at the hole itself; its diameter being to the diameter of the hole as 5 to 6, or as $5\frac{1}{2}$ to $6\frac{1}{2}$, very nearly, if I took the measures of those diameters right. I procured a very thin flat plate, having a hole pierced in the middle, the diameter of the circular hole being $\frac{5}{8}$ parts of an inch. And that the stream of running waters

might not be accelerated in falling, and by that acceleration become narrower, I fixed this plate not to the bottom, but to the side of the vessel, so as to make the water go out in the direction of a line parallel to the horizon. Then, when the vessel was full of water, I opened the hole to let it run out; and the diameter of the stream, measured with great accuracy at the distance of about half an inch from the hole, was $\frac{21}{25}$ of an inch. Therefore the diameter of this circular hole was to the diameter of the stream very nearly as 25 to 21. So that the water in passing through the hole converges on all sides, and, after it has run out of the vessel, becomes smaller by converging in that manner, and by becoming smaller is accelerated till it comes to the distance of half an inch from the hole, and at that distance flows in a smaller stream and with greater celerity than in the hole itself, and this in the ratio of 25×25 to 21×21 , or 17 to 12, very nearly; that is, in about the subduplicate ratio of 2 to 1. Now it is certain from experiments, that the quantity of water running out in a given time through a circular hole made in the bottom of a vessel is equal to the quantity which, flowing with the aforesaid velocity, would run out in the same time through another circular hole, whose diameter is to the diameter of the former as 21 to 25. And therefore that running water, in passing through the hole itself, has a velocity downwards equal to that which a heavy body would acquire in falling through half the height of the stagnant water in the vessel, nearly. But then, after it has run out, it is still accelerated by converging, till it arrives at a distance from the hole that is nearly equal to its diameter, and acquires a velocity greater than the other in about the subduplicate ratio of 2 to 1; which velocity a heavy body would nearly acquire by falling through the whole height of the stagnant water in the vessel.

Therefore in what follows let the diameter of the stream be represented by that lesser hole which we called EF. And imagine another plane VW above the hole EF (Pl. 7, Fig. 3), and parallel to the plane thereof, to be placed at a distance equal to the diameter of the same hole, and to be pierced through with a greater hole ST, of such a magnitude that

a stream which will exactly fill the lower hole EF may pass through it; the diameter of which hole will therefore be to the diameter of the lower hole as 25 to 21, nearly. By this means the water will run perpendicularly out at the lower hole; and the quantity of the water running out will be, according to the magnitude of this last hole, the same, very nearly, which the solution of the problem requires. The space included between the two planes and the falling stream may be considered as the bottom of the vessel. But, to make the solution more simple and mathematical, it is better to take the lower plane alone for the bottom of the vessel, and to suppose that the water which flowed through the ice as through a funnel, and ran out of the vessel through the hole EF made in the lower plane, preserves its motion continually, and that the ice continues at rest. Therefore in what follows let ST be the diameter of a circular hole described from the centre Z, and let the stream run out of the vessel through that hole, when the water in the vessel is all fluid. And let EF be the diameter of the hole, which the stream, in falling through, exactly fills up, whether the water runs out of the vessel by that upper hole ST, or flows through the middle of the ice in the vessel, as through a funnel. And let the diameter of the upper hole ST be to the diameter of the lower EF as about 25 to 21, and let the perpendicular distance between the planes of the holes be equal to the diameter of the lesser hole EF. Then the velocity of the water downwards, in running out of the vessel through the hole ST, will be in that hole the same that a body may acquire by falling from half the height IZ; and the velocity of both the falling streams will be in the hole EF, the same which a body would acquire by falling from the whole height IG.

CASE 2. If the hole EF be not in the middle of the bottom of the vessel, but in some other part thereof, the water will still run out with the same velocity as before, if the magnitude of the hole be the same. For though a heavy body takes a longer time in descending to the same depth, by an oblique line, than by a perpendicular line, yet in both cases it acquires in its descent the same velocity; as *Galileo* has demonstrated.

CASE 3. The velocity of the water is the same when it runs out through a hole in the side of the vessel. For if the hole be small, so that the interval between the superficies AB and KL may vanish as to sense, and the stream of water horizontally issuing out may form a parabolic figure; from the latus rectum of this parabola may be collected, that the velocity of the effluent water is that which a body may acquire by falling the height IG or HG of the stagnant water in the vessel. For, by making an experiment, I found that if the height of the stagnant water above the hole were 20 inches, and the height of the hole above a plane parallel to the horizon were also 20 inches, a stream of water springing out from thence would fall upon the plane, at the distance of 37 inches, very nearly, from a perpendicular let fall upon that plane from the hole. For without resistance the stream would have fallen upon the plane at the distance of 40 inches, the latus rectum of the parabolic stream being 80 inches.

CASE 4. If the effluent water tend upwards, it will still issue forth with the same velocity. For the small stream of water springing upwards, ascends with a perpendicular motion to GH or GI, the height of the stagnant water in the vessel; excepting in so far as its ascent is hindered a little by the resistance of the air; and therefore it springs out with the same velocity that it would acquire in falling from that height. Every particle of the stagnant water is equally pressed on all sides (by prop. 19, book 2), and, yielding to the pressure, tends all ways with an equal force, whether it descends through the hole in the bottom of the vessel, or gushes out in an horizontal direction through an hole in the side, or passes into a canal, and springs up from thence through a little hole made in the upper part of the canal. And it may not only be collected from reasoning, but is manifest also from the well-known experiments just mentioned, that the velocity with which the water runs out is the very same that is assigned in this proposition.

CASE 5. The velocity of the effluent water is the same, whether the figure of the hole be circular, or square, or triangular, or any other figure equal to the circular; for the ve-

locity of the effluent water does not depend upon the figure of the hole, but arises from its depth below the plane KL.

CASE 6. If the lower part of the vessel ABDC be immersed into stagnant water, and the height of the stagnant water above the bottom of the vessel be GR, the velocity with which the water that is in the vessel will run out at the hole EF into the stagnant water will be the same which the water would acquire by falling from the height IR; for the weight of all the water in the vessel that is below the superficies of the stagnant water will be sustained in equilibrio by the weight of the stagnant water, and therefore does not at all accelerate the motion of the descending water in the vessel. This case will also appear by experiments, measuring the times in which the water will run out.

COR. 1. Hence if CA, the depth of the water, be produced to K, so that AK may be to CK in the duplicate ratio of the area of a hole made in any part of the bottom to the area of the circle AB, the velocity of the effluent water will be equal to the velocity which the water would acquire by falling from the height KC.

COR. 2. And the force with which the whole motion of the effluent water may be generated is equal to the weight of a cylindric column of water, whose base is the hole EF, and its altitude 2GI or 2CK. For the effluent water, in the time it becomes equal to this column, may acquire, by falling by its own weight from the height GI, a velocity equal to that with which it runs out.

COR. 3. The weight of all the water in the vessel ABDC is to that part of the weight which is employed in forcing out the water as the sum of the circles AB and EF to twice the circle EF. For let IO be a mean proportional between IH and IG, and the water running out at the hole EF will, in the time that a drop falling from I would describe the altitude IG, become equal to a cylinder whose base is the circle EF and its altitude 2IG, that is, to a cylinder whose base is the circle AB, and whose altitude is 2IO. For the circle EF is to the circle AB in the subduplicate ratio of the altitude IH to the altitude IG; that is, in the simple ratio of the

mean proportional IO to the altitude IG. Moreover, in the time that a drop falling from I can describe the altitude IH, the water that runs out will have become equal to a cylinder whose base is the circle AB, and its altitude $2IH$; and in the time that a drop falling from I through H to G describes HG, the difference of the altitudes, the effluent water, that is, the water contained within the solid ABNFEM, will be equal to the difference of the cylinders, that is, to a cylinder whose base is AB, and its altitude $2HO$. And therefore all the water contained in the vessel ABDC is to the whole falling water contained in the said solid ABNFEM as HG to $2HO$, that is, as $HO + OG$ to $2HO$, or $IH + IO$ to $2IH$. But the weight of all the water in the solid ABNFEM is employed in forcing out the water; and therefore the weight of all the water in the vessel is to that part of the weight that is employed in forcing out the water as $IH + IO$ to $2IH$, and therefore as the sum of the circles EF and AB to twice the circle EF.

COR. 4. And hence the weight of all the water in the vessel ABDC is to the other part of the weight which is sustained by the bottom of the vessel as the sum of the circles AB and EF to the difference of the same circles.

COR. 5. And that part of the weight which the bottom of the vessel sustains is to the other part of the weight employed in forcing out the water as the difference of the circles AB and EF to twice the lesser circle EF, or as the area of the bottom to twice the hole.

COR. 6. That part of the weight which presses upon the bottom is to the whole weight of the water perpendicularly incumbent thereon as the circle AB to the sum of the circles AB and EF, or as the circle AB to the excess of twice the circle AB above the area of the bottom. For that part of the weight which presses upon the bottom is to the weight of the whole water in the vessel as the difference of the circles AB and EF to the sum of the same circles (by cor. 4); and the weight of the whole water in the vessel is to the weight of the whole water perpendicularly incumbent on the bottom as the circle AB to the difference of the circles AB and EF. There-

fore, *ex æquo perturbatè*, that part of the weight which presses upon the bottom is to the weight of the whole water perpendicularly incumbent thereon as the circle AB to the sums of the circles AB and EF, or the excess of twice the circle AB above the bottom.

COR. 7. If in the middle of the hole EF there be placed the little circle PQ described about the centre G, and parallel to the horizon, the weight of water which that little circle sustains is greater than the weight of a third part of a cylinder of water whose base is that little circle and its height GH. For let ABNFEM (Pl. 7, Fig. 4) be the cataract or column of falling water whose axis is GH, as above, and let all the water, whose fluidity is not requisite for the ready and quick descent of the water, be supposed to be congealed, as well round about the cataract, as above the little circle. And let PHQ be the column of water congealed above the little circle, whose vertex is H, and its altitude GH. And suppose this cataract to fall with its whole weight downwards, and not in the least to lie against or to press PHQ, but to glide freely by it without any friction, unless, perhaps, just at the very vertex of the ice, where the cataract at the beginning of its fall may tend to a concave figure. And as the congealed water AMEC, BNFD, lying round the cataract, is convex in its internal superficies AME, BNF, towards the falling cataract, so this column PHQ will be convex towards the cataract also, and will therefore be greater than a cone whose base is that little circle PQ and its altitude GH; that is, greater than a third part of a cylinder described with the same base and altitude. Now that little circle sustains the weight of this column, that is, a weight greater than the weight of the cone, or a third part of the cylinder.

COR. 8. The weight of water which the circle PQ, when very small, sustains, seems to be less than the weight of two thirds of a cylinder of water whose base is that little circle, and its altitude HG. For, things standing as above supposed, imagine the half of a spheroid described whose base is that little circle, and its semi-axis or altitude HG. This figure will be equal to two thirds of that cylinder, and will compre-

hend within it the column of congealed water PHQ, the weight of which is sustained by that little circle. For though the motion of the water tends directly downwards, the external superficies of that column must yet meet the base PQ in an angle somewhat acute, because the water in its fall is perpetually accelerated, and by reason of that acceleration become narrower. Therefore, since that angle is less than a right one, this column in the lower parts thereof will lie within the hemi-spheroid. In the upper parts also it will be acute or pointed; because, to make it otherwise, the horizontal motion of the water must be at the vertex infinitely more swift than its motion towards the horizon. And the less this circle PQ is, the more acute will the vertex of this column be; and the circle being diminished *in infinitum*, the angle PHQ will be diminished *in infinitum*, and therefore the column will lie within the hemi-spheroid. Therefore that column is less than that hemi-spheroid, or than two third parts of the cylinder whose base is that little circle, and its altitude GH. Now the little circle sustains a force of water equal to the weight of this column, the weight of the ambient water being employed in causing its efflux out at the hole.

COR. 9. The weight of water which the little circle PQ sustains, when it is very small, is very nearly equal to the weight of a cylinder of water whose base is that little circle, and its altitude $\frac{1}{2}$ GH; for this weight is an arithmetical mean between the weights of the cone and the hemi-spheroid abovementioned. But if that little circle be not very small, but on the contrary increased till it be equal to the hole EF, it will sustain the weight of all the water lying perpendicularly above it, that is, the weight of a cylinder of water whose base is that little circle, and its altitude GH.

COR. 10. And (as far as I can judge) the weight which this little circle sustains is always to the weight of a cylinder of water whose base is that little circle, and its altitude $\frac{1}{2}$ GH, as EF^2 to $EF^2 - \frac{1}{4}PQ^2$, or as the circle EF to the excess of this circle above half the little circle PQ, very nearly.

LEMMA IV.

If a cylinder move uniformly forwards in the direction of its length, the resistance made thereto is not at all changed by augmenting or diminishing that length; and is therefore the same with the resistance of a circle, described with the same diameter, and moving forwards with the same velocity in the direction of a right line perpendicular to its plane.

For the sides are not at all opposed to the motion; and a cylinder becomes a circle when its length is diminished in *infinitum*.

PROPOSITION XXXVII. THEOREM XXIX.

If a cylinder move uniformly forwards in a compressed, infinite, and non-elastic fluid, in the direction of its length, the resistance arising from the magnitude of its transverse section is to the force by which its whole motion may be destroyed or generated, in the time that it moves four times its length, as the density of the medium to the density of the cylinder, nearly.

For let the vessel ABDC (Pl. 7, Fig. 5) touch the surface of stagnant water with its bottom CD, and let the water run out of this vessel into the stagnant water through the cylindric canal EFTS perpendicular to the horizon; and let the little circle PQ be placed parallel to the horizon any where in the middle of the canal; and produce CA to K, so that AK may be to CK in the duplicate of the ratio, which the excess of the orifice of the canal EF above the little circle PQ bears to the circle AB. Then it is manifest (by case 5, case 6, and cor. 1, prop. 36) that the velocity of the water passing through the annular space between the little circle and the sides of the vessel will be the very same which the water would acquire by falling, and in its fall describing the altitude KC or IG.

And (by cor. 10, prop. 36) if the breadth of the vessel be infinite, so that the lineolæ HI may vanish, and the altitudes IG, HG, become equal; the force of the water that flows down and presses upon the circle will be to the weight of a cylinder whose base is that little circle, and the altitude $\frac{1}{2}IG$, as EF^2 to $EF^2 - \frac{1}{2}PQ^2$, very nearly. For the force of the water flowing downwards uniformly through the whole canal will be the same upon the little circle PQ in whatsoever part of the canal it be placed.

Let now the orifices of the canal EF, ST, be closed, and let the little circle ascend in the fluid compressed on every side, and by its ascent let it oblige the water that lies above it to descend through the annular space between the little circle and the sides of the canal. Then will the velocity of the ascending little circle be to the velocity of the descending water as the difference of the circles EF and PQ is to the circle PQ; and the velocity of the ascending little circle will be to the sum of the velocities, that is, to the relative velocity of the descending water with which it passes by the little circle in its ascent, as the difference of the circles EF and PQ to the circle EF, or as $EF^2 - PQ^2$ to EF^2 . Let that relative velocity be equal to the velocity with which it was shewn above that the water would pass through the annular space, if the circle were to remain unmoved, that is, to the velocity which the water would acquire by falling, and in its fall describing the altitude IG; and the force of the water upon the ascending circle will be the same as before (by cor. 5, of the laws of motion); that is, the resistance of the ascending little circle will be to the weight of a cylinder of water whose base is that little circle, and its altitude $\frac{1}{2}IG$, as EF^2 to $EF^2 - \frac{1}{2}PQ^2$, nearly. But the velocity of the little circle will be to the velocity which the water acquires by falling, and in its fall describing the altitude IG, as $EF^2 - PQ^2$ to EF^2 .

Let the breadth of the canal be increased *in infinitum*; and the ratios between $EF^2 - PQ^2$ and EF^2 , and between EF^2 and $EF^2 - \frac{1}{2}PQ^2$, will become at last ratios of equality. And therefore the velocity of the little circle will now be the same which the water would acquire in falling, and in its fall describing the altitude IG; and the resistance will become equal to the weight of a cylinder whose base is that little circle, and its altitude half the altitude IG, from which the cylinder must fall to acquire the velocity of the ascending circle; and with this velocity the cylinder in the time of its fall will describe four times its length. But the resistance of the cylinder moving forwards with this velocity in the direction of its length is the same with the resistance of the little circle (by lem. 4), and is therefore nearly equal to the force by which its

motion may be generated while it describes four times its length.

If the length of the cylinder be augmented or diminished, its motion, and the time in which it describes four times its length, will be augmented or diminished in the same ratio; and therefore the force by which the motion, so increased or diminished, may be destroyed or generated, will continue the same, because the time is increased or diminished in the same proportion; and therefore that force remains still equal to the resistance of the cylinder, because (by lem. 4) that resistance will also remain the same.

If the density of the cylinder be augmented or diminished, its motion, and the force by which its motion may be generated or destroyed in the same time, will be augmented or diminished in the same ratio. Therefore the resistance of any cylinder whatsoever will be to the force by which its whole motion may be generated or destroyed, in the time during which it moves four times its length, as the density of the medium to the density of the cylinder, nearly. Q.E.D.

A fluid must be compressed to become continued; it must be continued and non-elastic, that all the pressure arising from its compression may be propagated in an instant; and so, acting equally upon all parts of the body moved, may produce no change of the resistance. The pressure arising from the motion of the body is spent in generating a motion in the parts of the fluid, and this creates the resistance. But the pressure arising from the compression of the fluid, be it ever so forcible, if it be propagated in an instant, generates no motion in the parts of a continued fluid, produces no change at all of motion therein; and therefore neither augments nor lessens the resistance. This is certain, that the action of the fluid arising from the compression cannot be stronger on the hinder parts of the body moved than on its fore parts, and therefore cannot lessen the resistance described in this proposition. And if its propagation be infinitely swifter than the motion of the body pressed, it will not be stronger on the fore parts than on the hinder parts. But that action will be infinitely swifter, and propagated in an instant, if the fluid be continued and non-elastic.

COR. 1. The resistances made to cylinders going uniformly forwards in the direction of their lengths through continued infinite mediums, are in a ratio compounded of the duplicate ratio of the velocities and the duplicate ratio of the diameters, and the ratio of the density of the mediums.

COR. 2. If the breadth of the canal be not infinitely increased, but the cylinder go forwards in the direction of its length through an included quiescent medium, its axis all the while coinciding with the axis of the canal, its resistance will be to the force by which its whole motion, in the time in which it describes four times its length, may be generated or destroyed, in a ratio compounded of the ratio of EF^2 to $EF^2 - \frac{1}{2}PQ^2$ once, and the ratio of EF^2 to $EF^2 - PQ^2$ twice, and the ratio of the density of the medium to the density of the cylinder.

COR. 3. The same things supposed, and that a length L is to the quadruple of the length of the cylinder in a ratio compounded of the ratio $EF^2 - \frac{1}{2}PQ^2$ to EF^2 once, and the ratio of $EF^2 - PQ^2$ to EF^2 twice; the resistance of the cylinder will be to the force by which its whole motion, in the time during which it describes the length L , may be destroyed or generated, as the density of the medium to the density of the cylinder.

SCHOLIUM.

In this proposition we have investigated that resistance alone which arises from the magnitude of the transverse section of the cylinder, neglecting that part of the same which may arise from the obliquity of the motions. For as, in case 1 of prop. 36, the obliquity of the motions with which the parts of the water in the vessel converged on every side to the hole EF hindered the efflux of the water through the hole, so, in this proposition, the obliquity of the motions, with which the parts of the water, pressed by the antecedent extremity of the cylinder, yield to the pressure, and diverge on all sides, retards their passage through the places that lie round that antecedent extremity, towards the hinder parts of the cylinder, and causes the fluid to be moved to a greater distance; which increases the resistance, and that in the same ratio almost in which it diminished the efflux of the water out of the vessel,

that is, in the duplicate ratio of 25 to 21, nearly; and as, in case 1, of that proposition, we made the parts of the water pass through the hole EF perpendicularly and in the greatest plenty, by supposing all the water in the vessel lying round the cataract to be frozen, and that part of the water whose motion was oblique and useless to remain without motion, so in this proposition, that the obliquity of the motions may be taken away, and the parts of the water may give the freest passage to the cylinder, by yielding to it with the most direct and quick motion possible, so that only so much resistance may remain as arises from the magnitude of the transverse section, and which is incapable of diminution, unless by diminishing the diameter of the cylinder; we must conceive those parts of the fluid whose motions are oblique and useless, and produce resistance, to be at rest among themselves at both extremities of the cylinder, and there to cohere, and be joined to the cylinder. Let ABCD (Pl. 7, Fig. 6) be a rectangle, and let AE and BE be two parabolic arcs, described with the axis AB, and with a latus rectum that is to the space HG, which must be described by the cylinder in falling, in order to acquire the velocity with which it moves, as HG to $\frac{1}{2}$ AB. Let CF and DF be two other parabolic arcs described with the axis CD, and a latus rectum quadruple of the former; and by the convolution of the figure about the axis EF let there be generated a solid, whose middle part ABDC is the cylinder we are here speaking of, and whose extreme parts ABE and CDF contain the parts of the fluid at rest among themselves, and concreted into two hard bodies, adhering to the cylinder at each end like a head and tail. Then if this solid EACFDB move in the direction of the length of its axis FE towards the parts beyond E, the resistance will be the same which we have here determined in this proposition, nearly; that is, it will have the same ratio to the force with which the whole motion of the cylinder may be destroyed or generated, in the time that it is describing the length 4AC with that motion uniformly continued, as the density of the fluid has to the density of the cylinder, nearly. And (by cor.

7, prop. 36) the resistance must be to this force in the ratio of 2 to 3, at the least.

LEMMA V.

If a cylinder, a sphere, and a spheroid, of equal breadths be placed successively in the middle of a cylindric canal, so that their axes may coincide with the axis of the canal, these bodies will equally hinder the passage of the water through the canal.

For the spaces lying between the sides of the canal, and the cylinder, sphere, and spheroid, through which the water passes, are equal; and the water will pass equally through equal spaces.

This is true, upon the supposition that all the water above the cylinder, sphere, or spheroid, whose fluidity is not necessary to make the passage of the water the quickest possible, is congealed, as was explained above in cor. 7, prop. 36.

LEMMA VI.

The same supposition remaining, the forementioned bodies are equally acted on by the water flowing through the canal.

This appears by lem. 5, and the third law. For the water and the bodies act upon each other mutually and equally.

LEMMA VII.

If the water be at rest in the canal, and these bodies move with equal velocity and the contrary way through the canal, their resistances will be equal among themselves.

This appears from the last lemma, for the relative motions remain the same among themselves.

SCHOLIUM.

The case is the same of all convex and round bodies, whose axes coincide with the axis of the canal. Some difference may arise from a greater or less friction; but in these *lemmata* we suppose the bodies to be perfectly smooth, and the medium to be void of all tenacity and friction; and that those parts of the fluid which by their oblique and superfluous motions may disturb, hinder, and retard the flux of the water through the canal, are at rest amongst themselves; being fixed like water by frost, and adhering to the fore and hinder

parts of the bodies in the manner explained in the scholium of the last proposition; for in what follows we consider the very least resistance that round bodies described with the greatest given transverse sections can possibly meet with.

Bodies swimming upon fluids, when they move straight forwards, cause the fluid to ascend at their fore parts and subside at their hinder parts, especially if they are of an obtuse figure; and thence they meet with a little more resistance than if they were acute at the head and tail. And bodies moving in elastic fluids, if they are obtuse behind and before, condense the fluid a little more at their fore parts, and relax the same at their hinder parts; and therefore meet also with a little more resistance than if they were acute at the head and tail. But in these lemmas and propositions we are not treating of elastic but non-elastic fluids; not of bodies floating on the surface of the fluid, but deeply immersed therein. And when the resistance of bodies in non-elastic fluids is once known, we may then augment this resistance a little in elastic fluids, as our air; and in the surfaces of stagnating fluids, as lakes and seas.

PROPOSITION XXXVIII. THEOREM XXX.

If a globe move uniformly forward in a compressed, infinite, and non-elastic fluid, its resistance is to the force by which its whole motion may be destroyed or generated, in the time that it describes eight third parts of its diameter, as the density of the fluid to the density of the globe, very nearly.

For the globe is to its circumscribed cylinder as two to three; and therefore the force which can destroy all the motion of the cylinder while the same cylinder is describing the length of four of its diameters, will destroy all the motion of the globe while the globe is describing two thirds of this length, that is, eight third parts of its own diameter. Now the resistance of the cylinder is to this force very nearly as the density of the fluid to the density of the cylinder or globe (by prop. 37), and the resistance of the globe is equal to the resistance of the cylinder (by lem. 5, 6, and 7). Q.E.D.

COR. 1. The resistances of globes in infinite compressed mediums are in a ratio compounded of the duplicate ratio of the

velocity, and the duplicate ratio of the diameter, and the ratio of the density of the mediums.

COR. 2. The greatest velocity with which a globe can descend by its comparative weight through a resisting fluid, is the same which it may acquire by falling with the same weight, and without any resistance, and in its fall describing a space that is to four third parts of its diameter as the density of the globe to the density of the fluid. For the globe in the time of its fall, moving with the velocity acquired in falling, will describe a space that will be to eight third parts of its diameter as the density of the globe to the density of the fluid; and the force of its weight which generates this motion will be to the force that can generate the same motion, in the time that the globe describes eight third parts of its diameter, with the same velocity as the density of the fluid to the density of the globe; and therefore (by this proposition) the force of weight will be equal to the force of resistance, and therefore cannot accelerate the globe.

COR. 3. If there be given both the density of the globe and its velocity at the beginning of the motion, and the density of the compressed quiescent fluid in which the globe moves, there is given at any time both the velocity of the globe and its resistance, and the space described by it (by cor. 7, prop. 35).

COR. 4. A globe moving in a compressed quiescent fluid of the same density with itself will lose half its motion before it can describe the length of two of its diameters (by the same cor. 7).

PROPOSITION XXXIX. THEOREM XXXI.

If a globe move uniformly forward through a fluid inclosed and compressed in a cylindric canal, its resistance is to the force by which its whole motion may be generated or destroyed, in the time in which it describes eight third parts of its diameter, in a ratio compounded of the ratio of the orifice of the canal to the excess of that orifice above half the greatest circle of the globe; and the duplicate ratio of the orifice of the canal to the excess of that orifice above the

greatest circle of the globe; and the ratio of the density of the fluid to the density of the globe, nearly.

This appears by cor. 2, prop. 37, and the demonstration proceeds in the same manner as in the foregoing proposition.

SCHOLIUM.

In the two last propositions we suppose (as was done before in lem. 5) that all the water which precedes the globe, and whose fluidity increases the resistance of the same, is congealed. Now, if that water becomes fluid, it will somewhat increase the resistance. But in these propositions that increase is so small, that it may be neglected, because the convex superficies of the globe produces the very same effect almost as the congelation of the water.

PROPOSITION XL. PROBLEM IX.

To find by phaenomena the resistance of a globe moving through a perfectly fluid compressed medium.

Let A be the weight of the globe in *vacuo*, B its weight in the resisting medium, D the diameter of the globe, F a space which is to $\frac{1}{4}D$ as the density of the globe to the density of the medium, that is, as A to A — B, G the time in which the globe falling with the weight B without resistance describes the space F, and H the velocity which the body acquires by that fall. Then H will be the greatest velocity with which the globe can possibly descend with the weight B in the resisting medium, by cor. 2, prop. 38; and the resistance which the globe meets with, when descending with that velocity, will be equal to its weight B; and the resistance it meets with in any other velocity will be to the weight B in the duplicate ratio of that velocity to the greatest velocity H, by cor. 1, prop. 38.

This is the resistance that arises from the inactivity of the matter of the fluid. The resistance which arises from the elasticity, tenacity, and friction of its parts, may be thus investigated.

Let the globe be let fall so that it may descend in the fluid by the weight B; and let P be the time of falling, and let that time be expressed in seconds, if the time G be given in seconds. Find the absolute number N agreeing to the lo-

garithm $0,4342944819 \frac{2P}{G}$, and let L be the logarithm of

the number $\frac{N+1}{N}$; and the velocity acquired in falling will

be $\frac{N-1}{N+1} H$, and the height described will be $\frac{2PF}{G} -$

$1,3862943611F + 4,605170186LF$. If the fluid be of a sufficient depth, we may neglect the term $4,605170186LF$; and

$\frac{2PF}{G} - 1,3862943611F$ will be the altitude described, nearly.

These things appear by prop. 9, book 2, and its corollaries, and are true upon this supposition, that the globe meets with no other resistance but that which arises from the inactivity of matter. Now if it really meet with any resistance of another kind, the descent will be slower, and from the quantity of that retardation will be known the quantity of this new resistance.

That the velocity and descent of a body falling in a fluid might more easily be known, I have composed the following table; the first column of which denotes the time of descent; the second shews the velocities acquired in falling, the greatest velocity being 100000000; the third exhibits the spaces described by falling in those times, $2F$ being the space which the body describes in the time G with the greatest velocity; and the fourth gives the spaces described with the greatest velocity in the same times. The numbers in the fourth column

are $\frac{2P}{G}$, and by subducting the number $1,3862944 - 4,6051702L$,

are found the numbers in the third column; and these numbers must be multiplied by the space F to obtain the spaces described in falling. A fifth column is added to all these, containing the spaces described in the same times by a body falling *in vacuo* with the force of B its comparative weight.

The Times P.	Velocities of the body falling in the fluid.	The spaces describ- ed in falling in the fluid.	The spaces described with the greatest motion.	The spaces describ- ed by falling in <i>vacuo</i> .
0,001G	99999 $\frac{29}{36}$	0,000001F	0,002F	0,000001F
0,01G	999967	0,0001F	0,02F	0,0001F
0,1G	9966799	0,0099834F	0,2F	0,01F
0,2G	19737532	0,0397361F	0,4F	0,04F
0,3G	29131261	0,0886815F	0,6F	0,09F
0,4G	37994896	0,1559070F	0,8F	0,16F
0,5G	46211716	0,2402290F	1,0F	0,25F
0,6G	53704957	0,3402706F	1,2F	0,36F
0,7G	60436778	0,4545405F	1,4F	0,49F
0,8G	66403677	0,5815071F	1,6F	0,64F
0,9G	71629787	0,7196609F	1,8F	0,81F
1G	76159416	0,8675617F	2F	1F
2G	96402758	2,6500055F	4F	4F
3G	99505475	4,6186570F	6F	9F
4G	99932930	6,6143765F	8F	16F
5G	99990920	8,6137964F	10F	25F
6G	99998771	10,6137179F	12F	36F
7G	99999834	12,6137073F	14F	49F
8G	99999980	14,6137059F	16F	64F
9G	99999997	16,6137057F	18F	81F
10G	99999999 $\frac{1}{2}$	18,6137056F	20F	100F

SCHOLIUM.

In order to investigate the resistances of fluids from experiments, I procured a square wooden vessel, whose length and breadth on the inside was 9 inches *English* measure, and its depth 9 feet $\frac{1}{2}$; this I filled with rain-water; and having provided globes made up of wax, and lead included therein, I noted the times of the descents of these globes, the height through which they descended being 112 inches. A solid cubic foot of *English* measure contains 76 pounds *troy* weight of rain-water; and a solid inch contains $\frac{1}{2}$ ounces *troy* weight, or 253 $\frac{1}{2}$ grains; and a globe of water of one inch in diameter contains 132,645 grains in air, or 132,8 grains *in vacuo*; and any other globe will be as the excess of its weight *in vacuo* above its weight in water.

EXPER. 1. A globe whose weight was 156 $\frac{1}{2}$ grains in air, and 77 grains in water, described the whole height of 112

inches in 4 seconds. And, upon repeating the experiment, the globe spent again the very same time of 4 seconds in falling.

The weight of this globe, *in vacuo*, is $156\frac{1}{8}$ grains; and excess of this weight above the weight of the globe in water is $79\frac{1}{8}$ grains. Hence the diameter of the globe appears to be 0,84224 parts of an inch. Then it will be, as that excess to the weight of the globe *in vacuo*, so is the density of the water to the density of the globe; and so is $\frac{8}{9}$ parts of the diameter of the globe (*viz.* 2,24597 inches) to the space 2F, which will be therefore 4,4256 inches. Now a globe falling *in vacuo* with its whole weight of $156\frac{1}{8}$ grains in one second of time will describe $193\frac{1}{2}$ inches; and falling in water in the same time with the weight of 77 grains without resistance, will describe 95,219 inches; and in the time G, which is to one second of time in the subduplicate ratio of the space F, or of 2,2128 inches to 95,219 inches, will describe 2,2128 inches, and will acquire the greatest velocity H with which it is capable of descending in water. Therefore the time G is 0",15244. And in this time G, with that greatest velocity H, the globe will describe the space 2F, which is 4,4256 inches; and therefore in 4 seconds will describe a space of 116,1245 inches. Subtract the space 1,3862944F, or 3,0676 inches, and there will remain a space of 113,0569 inches, which the globe falling through water in a very wide vessel will describe in 4 seconds. But this space, by reason of the narrowness of the wooden vessel before-mentioned, ought to be diminished in a ratio compounded of the subduplicate ratio of the orifice of the vessel to the excess of this orifice above half a great circle of the globe, and of the simple ratio of the same orifice to its excess above a great circle of the globe, that is, in a ratio of 1 to 0,9914. This done, we have a space of 112,08 inches, which a globe falling through the water in this wooden vessel in 4 seconds of time ought nearly to describe by this theory; but it described 112 inches by the experiment.

EXPER. 2. Three equal globes, whose weights were severally $76\frac{1}{2}$ grains in air, and $5\frac{1}{16}$ grains in water, were let fall

ſucceſſively ; and every one fell through the water in 15 ſeconds of time, deſcribing in its fall a height of 112 inches.

By computation, the weight of each globe *in vacuo* is $76\frac{5}{12}$ grains ; the exceſs of this weight above the weight in water is 71 grains $\frac{17}{8}$; the diameter of the globe 0,81296 of an inch ; $\frac{8}{9}$ parts of this diameter 2,16789 inches ; the ſpace 2F is 2,3217 inches ; the ſpace which a globe of $5\frac{1}{16}$ grains in weight would deſcribe in one ſecond without reſiſtance, 12,808 inches, and the time G 0'',301056. Therefore the globe, with the greateſt velocity it is capable of receiving from a weight of $5\frac{1}{16}$ grains in its deſcent through water, will deſcribe in the time 0'',301056 the ſpace of 2,3217 inches ; and in 15 ſeconds the ſpace 115,678 inches. Subduct the ſpace 1,3862944F, or 1,609 inches, and there remains the ſpace 114,069 inches ; which therefore the falling globe ought to deſcribe in the ſame time, if the veſſel were very wide. But becauſe our veſſel was narrow, the ſpace ought to be diminiſhed by about 0,895 of an inch. And ſo the ſpace will remain 113,174 inches, which a globe falling in this veſſel ought nearly to deſcribe in 15 ſeconds, by the theory. But by the experiment it deſcribed 112 inches. The difference is not ſenſible.

EXPER. 3. Three equal globes, whoſe weights were ſeverally 121 grains in air, and 1 grain in water, were ſucceſſively let fall ; and they fell through the water in the times 46'', 47'', and 50'', deſcribing a height of 112 inches.

By the theory, theſe globes ought to have fallen in about 40''. Now whether their falling more ſlowly were occaſioned from hence, that in ſlow motions the reſiſtance ariſing from the force of inactivity does really bear a leſs proportion to the reſiſtance ariſing from other cauſes ; or whether it is to be attributed to little bubbles that might chance to ſtick to the globes, or to the rarefaction of the wax by the warmth of the weather, or of the hand that let them fall ; or, laſtly, whether it proceeded from ſome inſenſible errors in weighing the globes in the water, I am not certain. Therefore the weight of the globe in water ſhould be of ſeveral grains, that the experiment may be certain, and to be depended on.

EXPER. 4. I began the foregoing experiments to investigate the resistances of fluids, before I was acquainted with the theory laid down in the propositions immediately preceding. Afterwards, in order to examine the theory after it was discovered, I procured a wooden vessel, whose breadth on the inside was $8\frac{2}{3}$ inches, and its depth 15 feet and $\frac{1}{3}$. Then I made four globes of wax, with lead included, each of which weighed $139\frac{1}{4}$ grains in air, and $7\frac{1}{8}$ grains in water. These I let fall, measuring the times of their falling in the water with a pendulum oscillating to half seconds. The globes were cold, and had remained so some time, both when they were weighed and when they were let fall; because warmth rarefies the wax, and by rarefying it diminishes the weight of the globe in the water; and wax, when rarefied, is not instantly reduced by cold to its former density. Before they were let fall, they were totally immersed under water, lest, by the weight of any part of them that might chance to be above the water, their descent should be accelerated in its beginning. Then, when after their immersion they were perfectly at rest, they were let go with the greatest care, that they might not receive any impulse from the hand that let them down. And they fell successively in the times of $47\frac{1}{2}$, $48\frac{1}{2}$, 50, and 51 oscillations, describing a height of 15 feet and 2 inches. But the weather was now a little colder than when the globes were weighed, and therefore I repeated the experiment another day; and then the globes fell in the times of 49, $49\frac{1}{2}$, 50, and 53; and at a third trial in the times of $49\frac{1}{2}$, 50, 51, and 53 oscillations. And by making the experiment several times over, I found that the globes fell mostly in the times of $49\frac{1}{2}$ and 50 oscillations. When they fell slower, I suspect them to have been retarded by striking against the sides of the vessel.

Now, computing from the theory, the weight of the globe *in vacuo* is $139\frac{2}{3}$ grains; the excess of this weight above the weight of the globe in water $132\frac{11}{16}$ grains; the diameter of the globe 0,99868 of an inch; $\frac{8}{9}$ parts of the diameter 2,66315 inches; the space 2F, 2,8066 inches; the space which a globe weighing $7\frac{1}{8}$ grains falling without resistance describes in a se-

cond of time 9,88164 inches; and the time G, 0", 376843. Therefore the globe, with the greatest velocity with which it is capable of descending through the water by the force of a weight of $7\frac{1}{8}$ grains, will in the time 0", 376843 describe a space of 2,8066 inches, and in one second of time a space of 7,44766 inches, and in the time 25", or in 50 oscillations, the space 186,1915 inches. Subtract the space 1,386294F, or 1,9454 inches, and there will remain the space 184,2461 inches, which the globe will describe in that time in a very wide vessel. Because our vessel was narrow, let this space be diminished in a ratio compounded of the subduplicate ratio of the orifice of the vessel to the excess of this orifice above half a great circle of the globe, and of the simple ratio of the same orifice to its excess above a great circle of the globe; and we shall have the space of 181,86 inches, which the globe ought by the theory to describe in this vessel in the time of 50 oscillations, nearly. But it described the space of 182 inches, by experiment, in $49\frac{1}{2}$ or 50 oscillations.

EXPER. 5. Four globes, weighing $154\frac{3}{8}$ grains in air, and $21\frac{1}{2}$ grains in water, being let fall several times, fell in the times of $28\frac{1}{2}$, 29, $29\frac{1}{2}$, and 30, and sometimes of 31, 32, and 33 oscillations, describing a height of 15 feet and 2 inches.

They ought by the theory to have fallen in the time of 29 oscillations, nearly.

EXPER. 6. Five globes, weighing $212\frac{3}{8}$ grains in air, and $79\frac{1}{2}$ in water, being several times let fall, fell in the times of 15, $15\frac{1}{2}$, 16, 17, and 18 oscillations, describing a height of 15 feet and 2 inches.

By the theory they ought to have fallen in the time of 15 oscillations, nearly.

EXPER. 7. Four globes, weighing $293\frac{3}{8}$ grains in air, and $35\frac{7}{8}$ grains in water, being let fall several times, fell in the times of $29\frac{1}{2}$, 30, $30\frac{1}{2}$, 31, 32, and 33 oscillations, describing a height of 15 feet and 1 inch and $\frac{1}{2}$.

By the theory they ought to have fallen in the time of 28 oscillations, nearly.

In searching for the cause that occasioned these globes of the same weight and magnitude to fall, some swifter and

some slower, I hit upon this; that the globes, when they were first let go and began to fall, oscillated about their centres; that side which chanced to be the heavier descending first, and producing an oscillating motion. Now by oscillating thus, the globe communicates a greater motion to the water than if it descended without any oscillations; and by this communication loses part of its own motion with which it should descend; and therefore as this oscillation is greater or less, it will be more or less retarded. Besides, the globe always recedes from that side of itself which is descending in the oscillation, and by so receding comes nearer to the sides of the vessel, so as even to strike against them sometimes. And the heavier the globes are, the stronger this oscillation is; and the greater they are, the more is the water agitated by it. Therefore to diminish this oscillation of the globes, I made new ones of lead and wax, sticking the lead in one side of the globe very near its surface; and I let fall the globe in such a manner, that, as near as possible, the heavier side might be lowest at the beginning of the descent. By this means the oscillations became much less than before, and the times in which the globes fell were not so unequal: as in the following experiments.

EXPER. 8. Four globes, weighing 139 grains in air, and $6\frac{1}{2}$ in water, were let fall several times, and fell mostly in the time of 51 oscillations, never in more than 52, or in fewer than 50, describing a height of 182 inches.

By the theory they ought to fall in about the time of 52 oscillations.

EXPER. 9. Four globes weighing $273\frac{1}{4}$ grains in air, and $140\frac{3}{4}$ in water, being let fall several times, fell in never fewer than 12, and never more than 13 oscillations, describing a height of 182 inches.

These globes, by the theory, ought to have fallen in the time of $11\frac{1}{3}$ oscillations, nearly.

EXPER. 10. Four globes, weighing 384 grains in air, and $119\frac{1}{2}$ in water, being let fall several times, fell in the times of $17\frac{3}{4}$, 18, $18\frac{1}{2}$, and 19 oscillations, describing a height of $181\frac{1}{2}$ inches. And when they fell in the time of 19 oscillations, I

sometimes heard them hit against the sides of the vessel before they reached the bottom.

By the theory they ought to have fallen in the time of $15\frac{5}{9}$ oscillations, nearly.

EXPER. 11. Three equal globes, weighing 48 grains in the air, and $3\frac{2}{3}\frac{9}{2}$ in water, being several times let fall, fell in the times of $43\frac{1}{2}$, 44, $44\frac{1}{2}$, 45, and 46 oscillations, and mostly in 44 and 55, describing a height of $182\frac{1}{2}$ inches, nearly.

By the theory they ought to have fallen in the time of 46 oscillations and $\frac{5}{9}$, nearly.

EXPER. 12. Three equal globes, weighing 141 grains in air, and $4\frac{3}{8}$ in water, being let fall several times, fell in the times of 61, 62, 63, 64, and 65 oscillations, describing a space of 182 inches.

And by the theory they ought to have fallen in $64\frac{1}{2}$ oscillations, nearly.

From these experiments it is manifest, that when the globes fell slowly, as in the second, fourth, fifth, eighth, eleventh, and twelfth experiments, the times of falling are rightly exhibited by the theory; but when the globes fell more swiftly, as in the sixth, ninth, and tenth experiments, the resistance was somewhat greater than in the duplicate ratio of the velocity. For the globes in falling oscillate a little; and this oscillation, in those globes that are light and fall slowly, soon ceases by the weakness of the motion; but in greater and heavier globes, the motion being strong, it continues longer, and is not to be checked by the ambient water till after several oscillations. Besides, the more swiftly the globes move, the less are they pressed by the fluid at their hinder parts; and if the velocity be perpetually increased, they will at last leave an empty space behind them, unless the compression of the fluid be increased at the same time. For the compression of the fluid ought to be increased (by prop. 32 and 33) in the duplicate ratio of the velocity, in order to preserve the resistance in the same duplicate ratio. But because this is not done, the globes that move swiftly are not so much pressed at their hinder parts as the others; and by the defect of

this pressure it comes to pass that their resistance is a little greater than in a duplicate ratio of their velocity.

So that the theory agrees with the phenomena of bodies falling in water. It remains that we examine the phenomena of bodies falling in air.

EXPER. 13. From the top of St. Paul's Church in London, in June 1710, there were let fall together two glass globes, one full of quicksilver, the other of air; and in their fall they described a height of 220 *English* feet. A wooden table was suspended upon iron hinges on one side, and the other side of the same was supported by a wooden pin. The two globes lying upon this table were let fall together by pulling out the pin by means of an iron wire reaching from thence quite down to the ground; so that, the pin being removed, the table, which had then no support but the iron hinges, fell downwards, and, turning round upon the hinges, gave leave to the globes to drop off from it. At the same instant, with the same pull of the iron wire that took out the pin, a pendulum oscillating to seconds was let go, and began to oscillate. The diameters and weights of the globes, and their times of falling, are exhibited in the following table.

The globes filled with mercury.			The globes full of air.		
Weights.	Diameters.	Times in falling.	Weights.	Diameters.	Times in falling.
908 grains	0,8 of an inch	4"	510 grains	5,1 inches	8" $\frac{1}{2}$
983	0,8	4 —	642	5,2	8
866	0,8	4 —	599	5,1	8
747	0,75	4 +	515	5,0	8 $\frac{1}{4}$
808	0,75	4	483	5,0	8 $\frac{1}{2}$
784	0,75	4 +	641	5,2	8

But the times observed must be corrected; for the globes of mercury (by Galileo's theory), in 4 seconds of time, will describe 257 *English* feet, and 220 feet in only 3" 42". So that the wooden table, when the pin was taken out, did not turn upon its hinges so quickly as it ought to have done; and the slowness of that revolution hindered the descent of the globes at the beginning. For the globes lay about the middle of the table, and indeed were rather nearer to the axis upon which it turned than to the pin. And hence the times of falling

were prolonged about 18^{'''}; and therefore ought to be corrected by subducting that excess, especially in the larger globes, which, by reason of the largeness of their diameters, lay longer upon the revolving table than the others. This being done, the times in which the fix larger globes fell will come forth 8^{''} 12^{'''}, 7^{''} 42^{'''}, 7^{''} 42^{'''}, 7^{''} 57^{'''}, 8^{''} 12^{'''}, and 7^{''} 42^{'''}.

Therefore the fifth in order among the globes that were full of air being 5 inches in diameter, and 483 grains in weight, fell in 8^{''} 12^{'''}, describing a space of 220 feet. The weight of a bulk of water equal to this globe is 16600 grains; and the weight of an equal bulk of air is $\frac{16600}{880}$ grains, or 19 $\frac{3}{10}$ grains; and therefore the weight of the globe *in vacuo* is 502 $\frac{3}{10}$ grains; and this weight is to the weight of a bulk of air equal to the globe as 502 $\frac{3}{10}$ to 19 $\frac{3}{10}$; and so is 2F to $\frac{3}{8}$ of the diameter of the globe, that is, to 13 $\frac{1}{8}$ inches. Whence 2F becomes 28 feet 11 inches. A globe falling *in vacuo* with its whole weight of 502 $\frac{3}{10}$ grains, will in one second of time describe 193 $\frac{1}{8}$ inches as above; and with the weight of 483 grains will describe 185,905 inches; and with that weight 483 grains *in vacuo* will describe the space F, or 14 feet 5 $\frac{1}{2}$ inches, in the time of 57^{'''} 58^{'''}, and acquire the greatest velocity it is capable of descending with in the air. With this velocity the globe in 8^{''} 12^{'''} of time will describe 245 feet and 5 $\frac{1}{8}$ inches. Subtract 1,3863F, or 20 feet and $\frac{1}{2}$ an inch, and there remain 225 feet 5 inches. This space, therefore, the falling globe ought by the theory to describe in 8^{''} 12^{'''}; but by the experiment it described a space of 220 feet. The difference is insensible.

By like calculations applied to the other globes full of air, I composed the following table.

The weights of the globes.	The diameters.	The times of falling from a height of 220 feet.	The spaces which they would describe by the theory.	The excesses.
510 grains	5, 1 inches	8 ^{''} 12 ^{'''}	226 feet 11 inch.	6 feet 11 inch.
642	5, 2	7 42	230 9	10 9
599	5, 1	7 42	227 10	7 10
515	5	7 57	224 5	4 5
483	5	8 12	225 5	5 5
641	5, 2	7 42	230 7	10 7

EXPER. 14. *Anno* 1719, in the month of *July*, Dr. *Desaguliers* made some experiments of this kind again, by forming hogs' bladders into spherical orbs; which was done by means of a concave wooden sphere, which the bladders, being wetted well first, were put into. After that, being blown full of air, they were obliged to fill up the spherical cavity that contained them; and then, when dry, were taken out. These were let fall from the lantern on the top of the cupola of the same church, namely, from a height of 272 feet; and at the same moment of time there was let fall a leaden globe, whose weight was about 2 pounds *troy* weight. And in the mean time some persons standing in the upper part of the church where the globes were let fall observed the whole times of falling; and others standing on the ground observed the differences of the times between the fall of the leaden weight and the fall of the bladder. The times were measured by pendulums oscillating to half seconds. And one of those that stood upon the ground had a machine vibrating four times in one second; and another had another machine accurately made with a pendulum vibrating four times in a second also. One of those also who stood at the top of the church had a like machine; and these instruments were so contrived, that their motions could be stopped or renewed at pleasure. Now the leaden globe fell in about 4 seconds and $\frac{1}{4}$ of time; and from the addition of this time to the difference of time above spoken of, was collected the whole time in which the bladder was falling. The times which the five bladders spent in falling, after the leaden globe had reached the ground, were, the first time, $14\frac{3}{4}''$, $12\frac{3}{4}''$, $14\frac{5}{8}''$, $17\frac{3}{4}''$, and $16\frac{7}{8}''$, and the second time, $14\frac{1}{2}''$, $14\frac{1}{4}''$, $14''$, $19''$, and $16\frac{3}{4}''$. Add to these $4\frac{1}{4}''$, the time in which the leaden globe was falling, and the whole times in which the five bladders fell were, the first time, $19''$, $17''$, $18\frac{7}{8}''$, $22''$, and $21\frac{1}{8}''$; and the second time, $18\frac{3}{4}''$, $18\frac{1}{2}''$, $18\frac{1}{4}''$, $23\frac{1}{4}''$, and $21''$. The times observed at the top of the church were, the first time, $19\frac{3}{8}''$, $17\frac{1}{4}''$, $18\frac{3}{4}''$, $22\frac{1}{8}''$, and $21\frac{5}{8}''$; and the second time, $19''$, $18\frac{5}{8}''$, $18\frac{3}{8}''$, $24''$, and $21\frac{1}{4}''$. But the bladders did not always fall directly down, but sometimes fluttered a little in the air, and waved to and fro as they were descending. And by these motions the times of their falling were prolonged,

and increased by half a second sometimes, and sometimes by a whole second. The second and fourth bladder fell most directly the first time, and the first and third the second time. The fifth bladder was wrinkled, and by its wrinkles was a little retarded. I found their diameters by their circumferences measured with a very fine thread wound about them twice. In the following table I have compared the experiments with the theory; making the density of air to be to the density of rain-water as 1 to 860, and computing the spaces which by the theory the globes ought to describe in falling.

The weights of the bladders.	The diameters.	The times of falling from a height of 272 feet.	The spaces which by the theory ought to have been described in those times.	The difference between the theory and the experiments.
128 grains	5,28 inches	19"	271 feet 11 in.	—0ft. 1 in.
156	5,19	17	272 0 $\frac{1}{2}$	+ 0 0 $\frac{1}{2}$
137 $\frac{1}{2}$	5,3	18 $\frac{1}{2}$	272 7	+ 0 7
97 $\frac{1}{2}$	5,26	22	277 4	+ 5 4
99 $\frac{1}{8}$	5	21 $\frac{1}{8}$	282 0	+ 10 0

Our theory, therefore, exhibits rightly, within a very little, all the resistance that globes moving either in air or in water meet with; which appears to be proportional to the densities of the fluids in globes of equal velocities and magnitudes.

In the scholium subjoined to the sixth section, we shewed, by experiments of pendulums, that the resistances of equal and equally swift globes moving in air, water, and quicksilver, are as the densities of the fluids. We here prove the same more accurately by experiments of bodies falling in air and water. For pendulums at each oscillation excite a motion in the fluid always contrary to the motion of the pendulum in its return; and the resistance arising from this motion, as also the resistance of the thread by which the pendulum is suspended, makes the whole resistance of a pendulum greater than the resistance deduced from the experiments of falling bodies. For by the experiments of pendulums described in that scholium, a globe of the same density as water, in describing the length of its semi-diameter in air, would lose the $\frac{1}{3342}$ part of its motion. But by the theory delivered in this

seventh section, and confirmed by experiments of falling bodies, the same globe in describing the same length would lose only a part of its motion equal to $\frac{1}{4586}$, supposing the density of water to be to the density of air as 860 to 1. Therefore the resistances were found greater by the experiments of pendulums (for the reasons just mentioned) than by the experiments of falling globes; and that in the ratio of about 4 to 3. But yet since the resistances of pendulums oscillating in air, water, and quicksilver, are alike increased by like causes, the proportion of the resistances in these mediums will be rightly enough exhibited by the experiments of pendulums, as well as by the experiments of falling bodies. And from all this it may be concluded, that the resistances of bodies moving in any fluids whatsoever, though of the most extreme fluidity, are, *cæteris paribus*, as the densities of the fluids.

These things being thus established, we may now determine what part of its motion any globe projected in any fluid whatsoever would nearly lose in a given time. Let D be the diameter of the globe, and V its velocity at the beginning of its motion, and T the time in which a globe with the velocity V can describe *in vacuo* a space that is to the space $\frac{1}{2}D$ as the density of the globe to the density of the fluid; and the globe projected in that fluid will, in any other time t , lose the part

$\frac{tV}{T+t}$, the part $\frac{TV}{T+t}$ remaining; and will describe a space, which will be to that described in the same time *in vacuo* with the uniform velocity V , as the logarithm of the number $\frac{T+t}{T}$ multiplied by the number 2,302585093 is to the number

$\frac{t}{T}$, by cor. 7, prop. 35. In slow motions the resistance may be a little less, because the figure of a globe is more adapted to motion than the figure of a cylinder described with the same diameter. In swift motions the resistance may be a little greater, because the elasticity and compression of the fluid do not increase in the duplicate ratio of the velocity. But these little niceties I take no notice of.

And though air, water, quicksilver, and the like fluids, by the division of their parts *in infinitum*, should be subtilized, and become mediums infinitely fluid, nevertheless, the resistance they would make to projected globes would be the same. For the resistance considered in the preceding propositions arises from the inactivity of the matter; and the inactivity of matter is essential to bodies, and always proportional to the quantity of matter. By the division of the parts of the fluid the resistance arising from the tenacity and friction of the parts may be indeed diminished; but the quantity of matter will not be at all diminished by this division; and if the quantity of matter be the same, its force of inactivity will be the same; and therefore the resistance here spoken of will be the same, as being always proportional to that force. To diminish this resistance, the quantity of matter in the spaces through which the bodies move must be diminished; and therefore the celestial spaces, through which the globes of the planets and comets are perpetually passing towards all parts, with the utmost freedom, and without the least sensible diminution of their motion, must be utterly void of any corporeal fluid, excepting, perhaps, some extremely rare vapours and the rays of light.

Projectiles excite a motion in fluids as they pass through them; and this motion arises from the excess of the pressure of the fluid at the fore-parts of the projectile above the pressure of the same at the hinder parts; and cannot be less in mediums infinitely fluid than it is in air, water, and quicksilver, in proportion to the density of matter in each. Now this excess of pressure does, in proportion to its quantity, not only excite a motion in the fluid, but also acts upon the projectile so as to retard its motion; and therefore the resistance in every fluid is as the motion excited by the projectile in the fluid; and cannot be less in the most subtile æther in proportion to the density of that æther, than it is in air, water, and quicksilver, in proportion to the densities of those fluids.

SECTION VIII.

Of motion propagated through fluids.

PROPOSITION XLI. THEOREM XXXII.

A pressure is not propagated through a fluid in rectilinear directions, unless where the particles of the fluid lie in a right line. (Pl. 8, Fig. 1.)

If the particles a, b, c, d, e, lie in a right line, the pressure may be indeed directly propagated from a to e; but then the particle e will urge the obliquely posited particles f and g obliquely, and those particles f and g will not sustain this pressure, unless they be supported by the particles h and k lying beyond them; but the particles that support them are also pressed by them; and those particles cannot sustain that pressure, without being supported by, and pressing upon, those particles that lie still farther, as l and m, and so on *in infinitum*. Therefore the pressure, as soon as it is propagated to particles that lie out of right lines, begins to deflect towards one hand and the other, and will be propagated obliquely *in infinitum*; and after it has begun to be propagated obliquely, if it reaches more distant particles lying out of the right line, it will deflect again on each hand; and this it will do as often as it lights on particles that do not lie exactly in a right line. Q.E.D.

COR. If any part of a pressure, propagated through a fluid from a given point, be intercepted by any obstacle, the remaining part, which is not intercepted, will deflect into the spaces behind the obstacle. This may be demonstrated also after the following manner. Let a pressure be propagated from the point A (Pl. 8, Fig. 2) towards any part, and, if it be possible, in rectilinear directions; and the obstacle NBCK being perforated in BC, let all the pressure be intercepted but the coniform part APQ passing through the circular hole BC. Let the cone APQ be divided into frustums by the transverse planes de, fg, hi. Then while the cone ABC, propagating the pressure, urges the conic frustum degf beyond it on the superficies de, and this frustum urges the next frustum fgih on the superficies fg, and that frustum urges a third frustum, and so *in infinitum*; it is manifest (by the third law) that the

first frustum $defg$ is, by the re-action of the second frustum $fghi$, as much urged and pressed on the superficies fg , as it urges and presses that second frustum. Therefore the frustum $defg$ is compressed on both sides, that is, between the cone Ade and the frustum $fhig$; and therefore (by case 6, prop. 19) cannot preserve its figure, unless it be compressed with the same force on all sides. Therefore with the same force with which it is pressed on the superficies de , fg , it will endeavour to break forth at the sides df , eg ; and there (being not in the least tenacious or hard, but perfectly fluid) it will run out, expanding itself, unless there be an ambient fluid opposing that endeavour. Therefore, by the effort it makes to run out, it will press the ambient fluid, at its sides df , eg , with the same force that it does the frustum $fghi$; and therefore the pressure will be propagated as much from the sides df , eg , into the spaces NO , KL , this way and that way, as it is propagated from the superficies fg towards PQ . Q.E.D.

PROPOSITION XLII. THEOREM XXXIII.

All motion propagated through a fluid diverges from a rectilinear progress into the unmoved spaces. (Pl. 8, Fig. 3.)

CASE 1. Let a motion be propagated from the point A through the hole BC , and, if it be possible, let it proceed in the conic space $BCQP$ according to right lines diverging from the point A . And let us first suppose this motion to be that of waves in the surface of standing water; and let de , fg , hi , kl , &c. be the tops of the several waves, divided from each other by as many intermediate valleys or hollows. Then, because the water in the ridges of the waves is higher than in the unmoved parts of the fluid KL , NO , it will run down from off the tops of those ridges e , g , i , l , &c. d , f , h , k , &c. this way and that way towards KL and NO ; and because the water is more depressed in the hollows of the waves than in the unmoved parts of the fluid KL , NO , it will run down into those hollows out of those unmoved parts. By the first deflux the ridges of the waves will dilate themselves this way and that way; and be propagated towards KL and NO . And because the motion of the waves from A towards PQ is carried on by a continual deflux from the ridges of the waves into

the hollows next to them, and therefore cannot be swifter than in proportion to the celerity of the descent ; and the descent of the water on each side towards KL and NO must be performed with the same velocity ; it follows, that the dilatation of the waves on each side towards KL and NO will be propagated with the same velocity as the waves themselves go forward with directly from A to PQ. And therefore the whole space this way and that way towards KL and NO will be filled by the dilated waves *rfg*, *shis*, *tklt*, *vmnv*, &c. Q.E.D. That these things are so, any one may find by making the experiment in still water.

CASE 2. Let us suppose that *de*, *fg*, *hi*, *kl*, *mn*, represent pulses successively propagated from the point A through an elastic medium. Conceive the pulses to be propagated by successive condensations and rarefactions of the medium, so that the densest part of every pulse may occupy a spherical superficies described about the centre A, and that equal intervals intervene between the successive pulses. Let the lines *de*, *fg*, *hi*, *kl*, &c. represent the densest parts of the pulses, propagated through the hole BC ; and because the medium is denser there than in the spaces on either side towards KL and NO, it will dilate itself as well towards those spaces KL, NO, on each hand, as towards the rare intervals between the pulses ; and thence the medium, becoming always more rare next the intervals, and more dense next the pulses, will partake of their motion. And because the progressive motion of the pulses arises from the perpetual relaxation of the denser parts towards the antecedent rare intervals ; and since the pulses will relax themselves on each hand towards the quiescent parts of the medium KL, NO, with very near the same celerity ; therefore the pulses will dilate themselves on all sides into the unmoved parts KL, NO, with almost the same celerity with which they are propagated directly from the centre A, and therefore will fill up the whole space KLON. Q.E.D. And we find the same by experience also in sounds which are heard though a mountain interpose ; and, if they come into a chamber through the window, dilate themselves into all the parts of the room, and are heard in every corner ; and

not as reflected from the opposite walls, but directly propagated from the window, as far as our sense can judge.

CASE 3. Let us suppose, lastly, that a motion of any kind is propagated from A through the hole BC. Then since the cause of this propagation is that the parts of the medium that are near the centre A disturb and agitate those which lie farther from it; and since the parts which are urged are fluid, and therefore recede every way towards those spaces where they are less pressed, they will by consequence recede towards all the parts of the quiescent medium; as well to the parts on each hand, as KL and NO, as to those right before, as PQ; and by this means all the motion, as soon as it has passed through the hole BC, will begin to dilate itself, and from thence, as from its principle and centre, will be propagated directly every way. Q.E.D.

PROPOSITION XLIII. THEOREM XXXIV.

Every tremulous body in an elastic medium propagates the motion of the pulses on every side right forward; but in a non-elastic medium excites a circular motion:

CASE 1. The parts of the tremulous body alternately going and returning, do in going urge and drive before them those parts of the medium that lie nearest, and by that impulse compress and condense them; and in returning suffer those compressed parts to recede again, and expand themselves. Therefore the parts of the medium that lie nearest to the tremulous body move to and fro by turns, in like manner as the parts of the tremulous body itself do; and for the same cause that the parts of this body agitate these parts of the medium, these parts, being agitated by like tremors, will in their turn agitate others next to themselves; and these others, agitated in like manner, will agitate those that lie beyond them, and so on in infinitum. And in the same manner as the first parts of the medium were condensed in going, and relaxed in returning, so will the other parts be condensed every time they go, and expand themselves every time they return. And therefore they will not be all going and all returning at the same instant (for in that case they would always preserve determined

distances from each other, and there could be no alternate condensation and rarefaction); but since, in the places where they are condensed, they approach to, and, in the places where they are rarefied, recede from each other, therefore some of them will be going while others are returning; and so on *in infinitum*. The parts so going, and in their going condensed, are pulses, by reason of the progressive motion with which they strike obstacles in their way; and therefore the successive pulses produced by a tremulous body will be propagated in rectilinear directions; and that at nearly equal distances from each other, because of the equal intervals of time in which the body, by its several tremors, produces the several pulses. And though the parts of the tremulous body go and return in some certain and determinate direction, yet the pulses propagated from thence through the medium will dilate themselves towards the sides, by the foregoing proposition; and will be propagated on all sides from that tremulous body, as from a common centre, in superficies nearly spherical and concentrical. An example of this we have in waves excited by shaking a finger in water, which proceed not only forwards and backwards agreeably to the motion of the finger, but spread themselves in the manner of concentric circles all round the finger, and are propagated on every side. For the gravity of the water supplies the place of elastic force.

CASE 2. If the medium be not elastic, then, because its parts cannot be condensed by the pressure arising from the vibrating parts of the tremulous body, the motion will be propagated in an instant towards the parts where the medium yields most easily, that is, to the parts which the tremulous body would otherwise leave vacuum behind it. The case is the same with that of a body projected in any medium whatever. A medium yielding to projectiles does not recede *in infinitum*, but with a circular motion comes round to the spaces which the body leaves behind it. Therefore as often as a tremulous body tends to any part, the medium yielding to it comes round in a circle to the parts which the body leaves; and as often as the body returns to the first place, the me-

dium will be driven from the place it came round to, and return to its original place. And though the tremulous body be not firm and hard, but every way flexible, yet if it continue of a given magnitude, since it cannot impel the medium by its tremors any where without yielding to it somewhere else, the medium receding from the parts of the body where it is pressed will always come round in a circle to the parts that yield to it. Q.E.D.

COR. It is a mistake, therefore, to think, as some have done, that the agitation of the parts of flame conduces to the propagation of a pressure in rectilinear directions through an ambient medium. A pressure of that kind must be derived not from the agitation only of the parts of flame, but from the dilatation of the whole.

PROPOSITION XLIV. THEOREM XXXV.

If water ascend and descend alternately in the erected legs KL, MN, of a canal or pipe; and a pendulum be constructed whose length between the point of suspension and the centre of oscillation is equal to half the length of the water in the canal; I say, that the water will ascend and descend in the same times in which the pendulum oscillates. (Pl. 8, Fig. 4.)

I measure the length of the water along the axes of the canal and its legs, and make it equal to the sum of those axes; and take no notice of the resistance of the water arising from its attrition by the sides of the canal. Let, therefore, AB, CD, represent the mean height of the water in both legs; and when the water in the leg KL ascends to the height EF, the water will descend in the leg MN to the height GH. Let P be a pendulous body, VP the thread, V the point of suspension, RPQS the cycloid which the pendulum describes, P its lowest point, PQ an arc equal to the height AE. The force with which the motion of the water is accelerated and retarded alternately is the excess of the weight of the water in one leg above the weight in the other; and, therefore, when the water in the leg KL ascends to EF, and in the other leg descends to GH, that force is double the weight of the water EABF, and therefore is to the weight of the whole water as AE or PQ to VP or PR. The force also with which the body P is ac-

celerated or retarded in any place, as Q , of a cycloid, is (by cor. prop. 51) to its whole weight as its distance PQ from the lowest place P to the length PR of the cycloid. Therefore the motive forces of the water and pendulum, describing the equal spaces AE , PQ , are as the weights to be moved; and therefore if the water and pendulum are quiescent at first, those forces will move them in equal times, and will cause them to go and return together with a reciprocal motion. Q.E.D.

COR. 1. Therefore the reciprocations of the water in ascending and descending are all performed in equal times, whether the motion be more or less intense or remiss.

COR. 2. If the length of the whole water in the canal be of $6\frac{1}{2}$ feet of *French* measure, the water will descend in one second of time, and will ascend in another second, and so on by turns *in infinitum*; for a pendulum of $3\frac{1}{18}$ such feet in length will oscillate in one second of time.

COR. 3. But if the length of the water be increased or diminished, the time of the reciprocation will be increased or diminished in the subduplicate ratio of the length.

PROPOSITION XLV. THEOREM XXXVI.

The velocity of waves is in the subduplicate ratio of the breadths.

This follows from the construction of the following proposition.

PROPOSITION XLVI. PROBLEM X.

To find the velocity of waves.

Let a pendulum be constructed, whose length between the point of suspension and the centre of oscillation is equal to the breadth of the waves; and in the time that the pendulum will perform one single oscillation, the waves will advance forward nearly a space equal to their breadth.

That which I call the breadth of the waves is the transverse measure lying between the deepest part of the hollows, or the tops of the ridges. Let $ABCDEF$ (Pl. 8, Fig. 5) represent the surface of stagnant water ascending and descending in successive waves; and let A , C , E , &c. be the tops of the waves; and let B , D , F , &c. be the intermediate hollows.

Because the motion of the waves is carried on by the successive ascent and descent of the water, so that the parts thereof, as A, C, E, &c. which are highest at one time become lowest immediately after; and because the motive force, by which the highest parts descend and the lowest ascend, is the weight of the elevated water, that alternate ascent and descent will be analogous to the reciprocal motion of the water in the canal, and observe the same laws as to the times of its ascent and descent; and therefore (by prop. 44) if the distances between the highest places of the waves A, C, E, and the lowest B, D, F, be equal to twice the length of any pendulum, the highest parts A, C, E, will become the lowest in the time of one oscillation, and in the time of another oscillation will ascend again. Therefore between the passage of each wave the time of two oscillations will intervene; that is, the wave will describe its breadth in the time that pendulum will oscillate twice; but a pendulum of four times that length, and which therefore is equal to the breadth of the waves, will just oscillate once in that time. Q.E.I.

COR. 1. Therefore waves, whose breadth is equal to $3\frac{1}{2}$ French feet, will advance through a space equal to their breadth in one second of time; and therefore in one minute will go over a space of 183 $\frac{1}{2}$ feet; and in an hour a space of 11000 feet, nearly.

COR. 2. And the velocity of greater or less waves will be augmented or diminished in the subduplicate ratio of their breadth.

These things are true upon the supposition that the parts of water ascend or descend in a right line; but, in truth, that ascent and descent are rather performed in a circle; and therefore I propose the time defined by this proposition as only near the truth.

PROPOSITION XLVII. THEOREM XXXVII.

If pulses are propagated through a fluid, the several particles of the fluid, going and returning with the shortest reciprocal motion, are always accelerated or retarded according to the law of the oscillating pendulum. (Pl. 9, Fig. 1.)

Let AB, BC, CD, &c. represent equal distances of successive pulses; ABC the line of direction of the motion of the successive pulses propagated from A to B; E, F, G, three physical points of the quiescent medium situate in the right line AC at equal distances from each other; Ee, Ff, Gg, equal spaces of extreme shortness, through which those points go and return with a reciprocal motion in each vibration; ϵ , ϕ , γ , any intermediate places of the same points; EF, FG, physical lineolæ, or linear parts of the medium lying between those points, and successively transferred into the places $\epsilon\phi$, $\phi\gamma$, and $\epsilon\gamma$, fg. Let there be drawn the right line PS equal to the right line Ee. Bisection the same in O, and from the centre O, with the interval OP, describe the circle SIPI. Let the whole time of one vibration, with its proportional parts, be expounded by the whole circumference of this circle and its parts, in such sort, that, when any time PH or PHSh is completed, if there be let fall to PS the perpendicular HL or hl, and there be taken E ϵ equal to PL or Pl, the physical point E may be found in ϵ . A point, as E, moving according to this law with a reciprocal motion, in its going from E through ϵ to e, and returning again through ϵ to E, will perform its several vibrations with the same degrees of acceleration and retardation with those of an oscillating pendulum. We are now to prove that the several physical points of the medium will be agitated with such a kind of motion. Let us suppose, then, that a medium hath such a motion excited in it from any cause whatsoever, and consider what will follow from thence.

In the circumference PHSh let there be taken the equal arcs HI, IK, or hi, ik, having the same ratio to the whole circumference as the equal right lines EF, FG, have to BC, the whole interval of the pulses. Let fall the perpendiculars IM, KN, or im, kn; then because the points E, F, G, are successively agitated with like motions, and perform their entire vibrations composed of their going and return, while the pulse is transferred from B to C; if PH or PHSh be the time elapsed since the beginning of the motion of the point E, then will PI or PHSi be the time elapsed since the beginning of the

motion of the point F, and PK or PHSk the time elapsed since the beginning of the motion of the point G; and therefore $E\epsilon$, $F\phi$, $G\gamma$, will be respectively equal to PL, PM, PN, while the points are going, and to Pl, Pm, Pn, when the points are returning. Therefore $\epsilon\gamma$ or $EG + G\gamma - E\epsilon$ will, when the points are going, be equal to $EG - LN$, and in their return equal to $EG + Ln$. But $\epsilon\gamma$ is the breadth or expansion of the part EG of the medium in the place $\epsilon\gamma$; and therefore the expansion of that part in its going is to its mean expansion as $EG - LN$ to EG ; and in its return, as $EG + Ln$ or $EG + LN$ to EG . Therefore since LN is to KH as IM to the radius OP, and KH to EG as the circumference PHShP to BC; that is, if we put V for the radius of a circle whose circumference is equal to BC the interval of the pulses, as OP to V ; and, *ex æquo*, LN to EG as IM to V ; the expansion of the part EG, or of the physical point F in the place $\epsilon\gamma$, to the mean expansion of the same part in its first place EG, will be as $V - IM$ to V in going, and as $V + im$ to V in its return. Hence the elastic force of the point F in the place $\epsilon\gamma$ to its mean elastic force in the place EG is as $\frac{1}{V - IM}$ to $\frac{1}{V}$ in its going, and as

$\frac{1}{V + im}$ to $\frac{1}{V}$ in its return. And by the same reasoning the elastic forces of the physical points E and G in going are as $\frac{1}{V - HL}$ and $\frac{1}{V - KN}$ to $\frac{1}{V}$; and the difference of

the forces to the mean elastic force of the medium as $\frac{HL - KN}{VV - V \times HL - V \times KN + HL \times KN}$ to $\frac{1}{V}$; that is, as $\frac{HL - KN}{VV}$ to $\frac{1}{V}$, or as $HL - KN$ to V ; if we suppose (by reason of the

very short extent of the vibrations) HL and KN to be indefinitely less than the quantity V . Therefore since the quantity V is given, the difference of the forces is as $HL - KN$; that is (because $HL - KN$ is proportional to HK, and OM to OI or OP, and because HK and OP are given) as OM; that is, if Ff be bisected in Ω , as $\Omega\phi$. And for the same reason the difference of the elastic forces of the physical

points ϵ and γ , in the return of the physical lineolæ $\epsilon\gamma$, is as $\Omega\Phi$. But that difference (that is, the excess of the elastic force of the point ϵ above the elastic force of the point γ) is the very force by which the intervening physical lineolæ $\epsilon\gamma$ of the medium is accelerated in going, and retarded in returning; and therefore the accelerative force of the physical lineolæ $\epsilon\gamma$ is as its distance from Ω , the middle place of the vibration. Therefore (by prop. 38, book 1) the time is rightly expounded by the arc PI ; and the linear part of the medium $\epsilon\gamma$ is moved according to the law above-mentioned, that is, according to the law of a pendulum oscillating; and the case is the same of all the linear parts of which the whole medium is compounded. Q.E.D.

COR. Hence it appears that the number of the pulses propagated is the same with the number of the vibrations of the tremulous body, and is not multiplied in their progress. For the physical lineolæ $\epsilon\gamma$ as soon as it returns to its first place is at rest; neither will it move again, unless it receives a new motion either from the impulse of the tremulous body, or of the pulses propagated from that body. As soon, therefore, as the pulses cease to be propagated from the tremulous body, it will return to a state of rest, and move no more.

PROPOSITION XLVIII. THEOREM XXXVIII.

The velocities of pulses propagated in an elastic fluid are in a ratio compounded of the subduplicate ratio of the elastic force directly, and the subduplicate ratio of the density inversely; supposing the elastic force of the fluid to be proportional to its condensation.

CASE 1. If the mediums be homogeneous, and the distances of the pulses in those mediums be equal amongst themselves, but the motion in one medium is more intense than in the other, the contractions and dilatations of the correspondent parts will be as those motions: not that this proportion is perfectly accurate. However, if the contractions and dilatations are not exceedingly intense, the error will not be sensible; and therefore this proportion may be considered as physically exact. Now the motive elastic forces are as the contractions and dilatations; and the velocities generated in the same time in equal parts are as the forces.

Therefore equal and corresponding parts of corresponding pulses will go and return together, through spaces proportional to their contractions and dilatations, with velocities that are as those spaces; and therefore the pulses, which in the time of one going and returning advance forwards a space equal to their breadth, and are always succeeding into the places of the pulses that immediately go before them, will, by reason of the equality of the distances, go forward in both mediums with equal velocity.

CASE 2. If the distances of the pulses or their lengths are greater in one medium than in another, let us suppose that the correspondent parts describe spaces, in going and returning, each time proportional to the breadths of the pulses; then will their contractions and dilatations be equal; and therefore if the mediums are homogeneous, the motive elastic forces, which agitate them with a reciprocal motion, will be equal also. Now the matter to be moved by these forces is as the breadth of the pulses; and the space through which they move every time they go and return is in the same ratio. And, moreover, the time of one going and returning is in a ratio compounded of the subduplicate ratio of the matter, and the subduplicate ratio of the space; and therefore is as the space. But the pulses advance a space equal to their breadths in the times of going once and returning once; that is, they go over spaces proportional to the times, and therefore are equally swift.

CASE 3. And therefore in mediums of equal density and elastic force, all the pulses are equally swift. Now if the density or the elastic force of the medium were augmented, then, because the motive force is increased in the ratio of the elastic force, and the matter to be moved is increased in the ratio of the density, the time which is necessary for producing the same motion as before will be increased in the subduplicate ratio of the density, and will be diminished in the subduplicate ratio of the elastic force. And therefore the velocity of the pulses will be in a ratio compounded of the subduplicate ratio of the density of the medium inversely, and the subduplicate ratio of the elastic force directly. Q.E.D.

This proposition will be made more clear from the construction of the following problem.

PROPOSITION XLIX. PROBLEM XI.

The density and elastic force of a medium being given, to find the velocity of the pulses.

Suppose the medium to be pressed by an incumbent weight after the manner of our air; and let A be the height of a homogeneous medium, whose weight is equal to the incumbent weight, and whose density is the same with the density of the compressed medium in which the pulses are propagated. Suppose a pendulum to be constructed whose length between the point of suspension and the centre of oscillation is A : and in the time in which that pendulum will perform one entire oscillation composed of its going and returning, the pulse will be propagated right onwards through a space equal to the circumference of a circle described with the radius A .

For, letting these things stand which were constructed in prop. 47, if any physical line, as EF (Pl. 9, Fig. 1), describing the space PS in each vibration, be acted on in the extremities P and S of every going and return that it makes by an elastic force that is equal to its weight, it will perform its several vibrations in the time in which the same might oscillate in a cycloid whose whole perimeter is equal to the length PS ; and that because equal forces will impel equal corpuscles through equal spaces in the same or equal times. Therefore since the times of the oscillations are in the subduplicate ratio of the lengths of the pendulums, and the length of the pendulum is equal to half the arc of the whole cycloid, the time of one vibration would be to the time of the oscillation of a pendulum whose length is A in the subduplicate ratio of the length $\frac{1}{2}PS$ or PO to the length A . But the elastic force with which the physical lineolæ EG is urged, when it is found in its extreme places P, S , was (in the demonstration of prop. 47) to its whole elastic force as $HL - KN$ to V , that is (since the point K now falls upon P), as HK to V ; and all that force, or, which is the same thing, the incumbent weight by which the lineolæ EG is compressed, is to the weight of the lineolæ as the altitude A of the incumbent weight to EG the length

of the lineolæ; and therefore, *ex æquo*, the force with which the lineolæ EG is urged in the places P and S is to the weight of that lineolæ as $HK \times A$ to $V \times EG$; or as $PO \times A$ to VV ; because HK was to EG as PO to V. Therefore since the times in which equal bodies are impelled through equal spaces are reciprocally in the subduplicate ratio of the forces, the time of one vibration, produced by the action of that elastic force, will be to the time of a vibration produced by the impulse of the weight in a subduplicate ratio of VV to $PO \times A$, and therefore to the time of the oscillation of a pendulum whose length is A in the subduplicate ratio of VV to $PO \times A$, and the subduplicate ratio of PO to A conjunctly; that is, in the entire ratio of V to A. But in the time of one vibration composed of the going and returning of the pendulum, the pulse will be propagated right onwards through a space equal to its breadth BC. Therefore the time in which a pulse runs over the space BC is to the time of one oscillation composed of the going and returning of the pendulum as V to A, that is, as BC to the circumference of a circle whose radius is A. But the time in which the pulse will run over the space BC is to the time in which it will run over a length equal to that circumference in the same ratio; and therefore in the time of such an oscillation the pulse will run over a length equal to that circumference. Q.E.D.

COR. 1. The velocity of the pulses is equal to that which heavy bodies acquire by falling with an equally accelerated motion, and in their fall describing half the altitude A. For the pulse will, in the time of this fall, supposing it to move with the velocity acquired by that fall, run over a space that will be equal to the whole altitude A; and therefore in the time of one oscillation composed of one going and return, will go over a space equal to the circumference of a circle described with the radius A; for the time of the fall is to the time of oscillation as the radius of a circle to its circumference.

COR. 2. Therefore since that altitude A is as the elastic force of the fluid directly, and the density of the same inversely, the velocity of the pulses will be in a ratio compounded

of the subduplicate ratio of the density inversely, and the subduplicate ratio of the elastic force directly.

PROPOSITION L. PROBLEM XII.

To find the distances of the pulses.

Let the number of the vibrations of the body, by whose tremor the pulses are produced, be found to any given time. By that number divide the space which a pulse can go over in the same time, and the part found will be the breadth of one pulse. Q.E.I.

SCHOLIUM.

The last propositions respect the motions of light and sounds; for since light is propagated in right lines, it is certain that it cannot consist in action alone (by prop. 41 and 42). As to sounds, since they arise from tremulous bodies, they can be nothing else but pulses of the air propagated through it (by prop. 43); and this is confirmed by the tremors which sounds, if they be loud and deep, excite in the bodies near them, as we experience in the sound of drums; for quick and short tremors are less easily excited. But it is well known that any sounds, falling upon strings in unison with the sonorous bodies, excite tremors in those strings. This is also confirmed from the velocity of sounds; for since the specific gravities of rain-water and quicksilver are to one another as about 1 to $13\frac{2}{3}$, and when the mercury in the barometer is at the height of 30 inches of our measure, the specific gravities of the air and of rain-water are to one another as about 1 to 870, therefore the specific gravity of air and quicksilver are to each other as 1 to 11890. Therefore, when the height of the quicksilver is at 30 inches, a height of uniform air, whose weight would be sufficient to compress our air to the density we find it to be of, must be equal to 356700 inches, or 29725 feet of our measure: and this is that very height of the medium, which I have called A in the construction of the foregoing proposition. A circle whose radius is 29725 feet is 186768 feet in circumference. And since a pendulum $39\frac{1}{4}$ inches in length completes one oscillation, composed of its going and return, in two seconds of time, as is commonly known, it follows that a pendulum

29725 feet, or 356700 inches in length will perform a like oscillation in $190\frac{1}{4}$ seconds. Therefore in that time a sound will go right onwards 186768 feet, and therefore in one second 979 feet.

But in this computation we have made no allowance for the crassitude of the solid particles of the air, by which the sound is propagated instantaneously. Because the weight of air is to the weight of water as 1 to 870, and because salts are almost twice as dense as water; if the particles of air are supposed to be of near the same density as those of water or salt, and the rarity of the air arises from the intervals of the particles; the diameter of one particle of air will be to the interval between the centres of the particles as 1 to about 9 or 10, and to the interval between the particles themselves as 1 to 8 or 9. Therefore to 979 feet, which, according to the above calculation, a sound will advance forward in one second of time, we may add $97\frac{9}{10}$, or about 109 feet, to compensate for the crassitude of the particles of the air: and then a sound will go forward about 1088 feet in one second of time.

Moreover, the vapours floating in the air being of another spring, and a different tone, will hardly, if at all, partake of the motion of the true air in which the sounds are propagated. Now if these vapours remain unmoved, that motion will be propagated the swifter through the air alone, and that in the subduplicate ratio of the defect of the matter. So if the atmosphere consist of ten parts of true air and one part of vapours, the motion of sounds will be swifter in the subduplicate ratio of 11 to 10, or very nearly in the entire ratio of 21 to 20, than if it were propagated through eleven parts of true air; and therefore the motion of sounds above discovered must be increased in that ratio. By this means the sound will pass through 1142 feet in one second of time.

These things will be found true in spring and autumn, when the air is rarefied by the gentle warmth of those seasons, and by that means its elastic force becomes somewhat more intense. But in winter, when the air is condensed by the cold, and its elastic force is somewhat remitted, the motion

of sounds will be slower in a subduplicate ratio of the density; and, on the other hand, swifter in the summer.

Now by experiments it actually appears that sounds do really advance in one second of time about 1142 feet of *English* measure, or 1070 feet of *French* measure.

The velocity of sounds being known, the intervals of the pulses are known also. For M. *Sauveur*, by some experiments that he made, found that an open pipe about 5 *Paris* feet in length gives a sound of the same tone with a viol-string that vibrates a hundred times in one second. Therefore there are near 100 pulses in a space of 1070 *Paris* feet, which a sound runs over in a second of time; and therefore one pulse fills up a space of about $10\frac{7}{10}$ *Paris* feet, that is, about twice the length of the pipe. From whence it is probable that the breadths of the pulses, in all sounds made in open pipes, are equal to twice the length of the pipes.

Moreover, from the corollary of prop. 47 appears the reason why the sounds immediately cease with the motion of the sonorous body, and why they are heard no longer when we are at a great distance from the sonorous bodies than when we are very near them. And besides, from the foregoing principles, it plainly appears how it comes to pass that sounds are so mightily increased in speaking-trumpets; for all reciprocal motion uses to be increased by the generating cause at each return. And in tubes hindering the dilatation of the sounds, the motion decays more slowly, and recurs more forcibly; and therefore is the more increased by the new motion impressed at each return. And these are the principal phænomena of sounds.

SECTION IX.

Of the circular motion of fluids.

HYPOTHESIS.

The resistance arising from the want of lubricity in the parts of a fluid, is, cæteris paribus, proportional to the velocity with which the parts of the fluid are separated from each other.

PROPOSITION LI. THEOREM XXXVIII.

If a solid cylinder infinitely long, in an uniform and infinite fluid, revolve with an uniform motion about an axis given in position, and the fluid be forced round by only this impulse of the cylinder, and every part of the fluid persevere uniformly in its motion; I say, that the periodic times of the parts of the fluid are as their distances from the axis of the cylinder.

Let AFL (Pl. 9, Fig. 2) be a cylinder turning uniformly about the axis S, and let the concentric circles BGM, CHN, DIO, EKP, &c. divide the fluid into innumerable concentric cylindric solid orbs of the same thickness. Then, because the fluid is homogeneous, the impressions which the contiguous orbs make upon each other mutually will be (by the hypothesis) as their translations from each other, and as the contiguous superficies upon which the impressions are made. If the impression made upon any orb be greater or less on its concave than on its convex side, the stronger impression will prevail, and will either accelerate or retard the motion of the orb, according as it agrees with, or is contrary to, the motion of the same. Therefore, that every orb may persevere uniformly in its motion, the impressions made on both sides must be equal, and their directions contrary. Therefore since the impressions are as the contiguous superficies, and as their translations from one another, the translations will be inversely as the superficies, that is, inversely as the distances of the superficies from the axis. But the differences of the angular motions about the axis are as those translations applied to the distances, or as the translations directly and the distances inversely; that is, joining these ratios together, as the squares of the distances inversely. Therefore if there be erected the lines Aa, Bb, Cc, Dd, Ee, &c. perpendicular to the several parts of the infinite right line SABCDEQ, and reciprocally proportional to the squares of SA, SB, SC, SD, SE, &c. and through the extremities of those perpendiculars there be supposed to pass an hyperbolic curve, the sums of the differences, that is, the whole angular motions, will be as the correspondent sums of the lines Aa, Bb, Cc, Dd, Ee, that is

(if to constitute a medium uniformly fluid the number of the orbs be increased and their breadth diminished *in infinitum*), as the hyperbolic areas AaQ , BbQ , CcQ , DdQ , EeQ , &c. analogous to the fums; and the times, reciprocally proportional to the angular motions, will be also reciprocally proportional to those areas. Therefore the periodic time of any particle, as D , is reciprocally as the area DdQ , that is (as appears from the known methods of quadratures of curves), directly as the distance SD . Q.E.D.

COR. 1. Hence the angular motions of the particles of the fluid are reciprocally as their distances from the axis of the cylinder, and the absolute velocities are equal.

COR. 2. If a fluid be contained in a cylindric vessel of an infinite length, and contain another cylinder within, and both the cylinders revolve about one common axis, and the times of their revolutions be as their semi-diameters, and every part of the fluid perseveres in its motion, the periodic times of the several parts will be as the distances from the axis of the cylinders.

COR. 3. If there be added or taken away any common quantity of angular motion from the cylinder and fluid moving in this manner; yet, because this new motion will not alter the mutual attrition of the parts of the fluid, the motion of the parts among themselves will not be changed; for the translations of the parts from one another depend upon the attrition. Any part will persevere in that motion, which, by the attrition made on both sides with contrary directions, is no more accelerated than it is retarded.

COR. 4. Therefore if there be taken away from this whole system of the cylinders and the fluid all the angular motion of the outward cylinder, we shall have the motion of the fluid in a quiescent cylinder.

COR. 5. Therefore if the fluid and outward cylinder are at rest, and the inward cylinder revolve uniformly, there will be communicated a circular motion to the fluid, which will be propagated by degrees through the whole fluid; and will go on continually increasing, till such time as the several parts of the fluid acquire the motion determined in cor. 4.

COR. 6. And because the fluid endeavours to propagate its motion still farther, its impulse will carry the outmost cylinder also about with it, unless the cylinder be violently detained; and accelerate its motion till the periodic times of both cylinders become equal among themselves. But if the outward cylinder be violently detained, it will make an effort to retard the motion of the fluid; and unless the inward cylinder preserve that motion by means of some external force impressed thereon, it will make it cease by degrees.

All these things will be found true by making the experiment in deep standing water.

PROPOSITION LII. THEOREM XL.

If a solid sphere, in an uniform and infinite fluid, revolves about an axis given in position with an uniform motion, and the fluid be forced round by only this impulse of the sphere; and every part of the fluid perseveres uniformly in its motion: I say, that the periodic times of the parts of the fluid are as the squares of their distances from the centre of the sphere.

CASE 1. Let AFL be a sphere turning uniformly about the axis S, and let the concentric circles BGM, CHN, DIO, EKP, &c. divide the fluid into innumerable concentric orbs of the same thickness. Suppose those orbs to be solid; and, because the fluid is homogeneous, the impressions which the contiguous orbs make one upon another will be (by the supposition) as their translations from one another, and the contiguous superficies upon which the impressions are made. If the impression upon any orb be greater or less upon its concave than upon its convex side, the more forcible impression will prevail, and will either accelerate or retard the velocity of the orb, according as it is directed with a conspiring or contrary motion to that of the orb. Therefore that every orb may persevere uniformly in its motion, it is necessary that the impressions made upon both sides of the orb should be equal, and have contrary directions. Therefore since the impressions are as the contiguous superficies, and as their translations from one another, the translations will be inversely as the superficies, that is, inversely as the squares of the distances of the superficies from the centre. But the differences of the

angular motions about the axis are as those translations applied to the distances, or as the translations directly and the distances inversely; that is, by compounding those ratios, as the cubes of the distances inversely. Therefore if upon the several parts of the infinite right line $SABCDEQ$ there be erected the perpendiculars Aa, Bb, Cc, Dd, Ee , &c. reciprocally proportional to the cubes of SA, SB, SC, SD, SE , &c. the sums of the differences, that is, the whole angular motions, will be as the corresponding sums of the lines Aa, Bb, Cc, Dd, Ee , &c. that is (if to constitute an uniformly fluid medium the number of the orbs be increased and their thickness diminished *in infinitum*), as the hyperbolic areas AaQ, BbQ, CcQ, DdQ, EeQ , &c. analogous to the sums; and the periodic times being reciprocally proportional to the angular motions, will be also reciprocally proportional to those areas. Therefore the periodic time of any orb DIO is reciprocally as the area DdQ , that is (by the known methods of quadratures), directly as the square of the distance SD . Which was first to be demonstrated.

CASE 2. From the centre of the sphere let there be drawn a great number of indefinite right lines, making given angles with the axis, exceeding one another by equal differences; and, by these lines revolving about the axis, conceive the orbs to be cut into innumerable annuli; then will every annulus have four annuli contiguous to it, that is, one on its inside, one on its outside, and two on each hand. Now each of these annuli cannot be impelled equally and with contrary directions by the attrition of the interior and exterior annuli, unless the motion be communicated according to the law which we demonstrated in case 1. This appears from that demonstration. And therefore any series of annuli, taken in any right line extending itself *in infinitum* from the globe, will move according to the law of case 1, except we should imagine it hindered by the attrition of the annuli on each side of it. But now in a motion, according to this law, no such attrition is, and therefore cannot be, any obstacle to the motions persevering according to that law. If annuli at equal distances from the centre revolve either more swiftly or more

slowly near the poles than near the ecliptic, they will be accelerated if slow, and retarded if swift, by their mutual attrition; and so the periodic times will continually approach to equality, according to the law of case 1. Therefore this attrition will not at all hinder the motion from going on according to the law of case 1, and therefore that law will take place; that is, the periodic times of the several annuli will be as the squares of their distances from the centre of the globe. Which was to be demonstrated in the second place.

CASE 3. Let now every annulus be divided by transverse sections into innumerable particles constituting a substance absolutely and uniformly fluid; and because these sections do not at all respect the law of circular motion, but only serve to produce a fluid substance, the law of circular motion will continue the same as before. All the very small annuli will either not at all change their asperity and force of mutual attrition upon account of these sections, or else they will change the same equally. Therefore the proportion of the causes remaining the same, the proportion of the effects will remain the same also; that is, the proportion of the motions and the periodic times. Q.E.D. But now as the circular motion, and the centrifugal force thence arising, is greater at the ecliptic than at the poles, there must be some cause operating to retain the several particles in their circles; otherwise the matter that is at the ecliptic will always recede from the centre, and come round about to the poles by the outside of the vortex, and from thence return by the axis to the ecliptic with a perpetual circulation.

COR. 1. Hence the angular motions of the parts of the fluid about the axis of the globe are reciprocally as the squares of the distances from the centre of the globe, and the absolute velocities are reciprocally as the same squares applied to the distances from the axis.

COR. 2. If a globe revolve with a uniform motion about an axis of a given position in a similar and infinite quiescent fluid with an uniform motion, it will communicate a whirling motion to the fluid like that of a vortex, and that motion will by degrees be propagated onwards *in infinitum*; and this mo-

tion will be increased continually in every part of the fluid, till the periodical times of the several parts become as the squares of the distances from the centre of the globe.

COR. 3. Because the inward parts of the vortex are, by reason of their greater velocity, continually pressing upon and driving forwards the external parts, and by that action are perpetually communicating motion to them, and at the same time those exterior parts communicate the same quantity of motion to those that lie still beyond them, and by this action preserve the quantity of their motion continually unchanged, it is plain that the motion is perpetually transferred from the centre to the circumference of the vortex, till it is quite swallowed up and lost in the boundless extent of that circumference. The matter between any two spherical superficies concentric to the vortex will never be accelerated; because that matter will be always transferring the motion it receives from the matter nearer the centre to that matter which lies nearer the circumference.

COR. 4. Therefore, in order to continue a vortex in the same state of motion, some active principle is required from which the globe may receive continually the same quantity of motion which it is always communicating to the matter of the vortex. Without such a principle it will undoubtedly come to pass that the globe and the inward parts of the vortex, being always propagating their motion to the outward parts, and not receiving any new motion, will gradually move slower and slower, and at last be carried round no longer.

COR. 5. If another globe should be swimming in the same vortex at a certain distance from its centre, and in the mean time by some force revolve constantly about an axis of a given inclination, the motion of this globe will drive the fluid round after the manner of a vortex; and at first this new and small vortex will revolve with its globe about the centre of the other; and in the mean time its motion will creep on farther and farther, and by degrees be propagated in *infinitum*, after the manner of the first vortex. And for the same reason that the globe of the new vortex was carried about before by the motion of the other vortex, the globe of this

other will be carried about by the motion of this new vortex, so that the two globes will revolve about some intermediate point, and by reason of that circular motion mutually fly from each other, unless some force restrains them. Afterwards, if the constantly impressed forces, by which the globes persevere in their motions, should cease, and every thing be left to act according to the laws of mechanics, the motion of the globes will languish by degrees (for the reason assigned in cor. 3 and 4), and the vortices at last will quite stand still.

COR. 6. If several globes in given places should constantly revolve with determined velocities about axes given in position, there would arise from them as many vortices going on *in infinitum*. For upon the same account that any one globe propagates its motion *in infinitum*, each globe apart will propagate its own motion *in infinitum* also; so that every part of the infinite fluid will be agitated with a motion resulting from the actions of all the globes. Therefore the vortices will not be confined by any certain limits, but by degrees run mutually into each other; and by the mutual actions of the vortices on each other, the globes will be perpetually moved from their places, as was shewn in the last corollary; neither can they possibly keep any certain position among themselves, unless some force restrains them. But if those forces, which are constantly impressed upon the globes to continue these motions, should cease, the matter (for the reason assigned in cor. 3 and 4) will gradually stop, and cease to move in vortices.

COR. 7. If a similar fluid be inclosed in a spherical vessel, and, by the uniform rotation of a globe in its centre, is driven round in a vortex; and the globe and vessel revolve the same way about the same axis, and their periodical times be as the squares of the semi-diameters; the parts of the fluid will not go on in their motions without acceleration or retardation, till their periodical times are as the squares of their distances from the centre of the vortex. No constitution of a vortex can be permanent but this.

COR. 8. If the vessel, the inclosed fluid, and the globe, retain this motion, and revolve besides with a common angu-

lar motion about any given axis, because the mutual attrition of the parts of the fluid is not changed by this motion, the motions of the parts among each other will not be changed; for the translations of the parts among themselves depend upon this attrition. Any part will persevere in that motion in which its attrition on one side retards it just as much as its attrition on the other side accelerates it.

COR. 9. Therefore if the vessel be quiescent, and the motion of the globe be given, the motion of the fluid will be given. For conceive a plane to pass through the axis of the globe, and to revolve with a contrary motion; and suppose the sum of the time of this revolution and of the revolution of the globe to be to the time of the revolution of the globe as the square of the semi-diameter of the vessel to the square of the semi-diameter of the globe; and the periodic times of the parts of the fluid in respect of this plane will be as the squares of their distances from the centre of the globe.

COR. 10. Therefore if the vessel move about the same axis with the globe, or with a given velocity about a different one, the motion of the fluid will be given. For if from the whole system we take away the angular motion of the vessel, all the motions will remain the same among themselves as before, by cor. 8, and those motions will be given by cor. 9.

COR. 11. If the vessel and the fluid are quiescent, and the globe revolves with an uniform motion, that motion will be propagated by degrees through the whole fluid to the vessel, and the vessel will be carried round by it, unless violently detained; and the fluid and the vessel will be continually accelerated till their periodic times become equal to the periodic times of the globe. If the vessel be either withheld by some force, or revolve with any constant and uniform motion, the medium will come by little and little to the state of motion defined in cor. 8, 9, 10, nor will it ever persevere in any other state. But if then the forces, by which the globe and vessel revolve with certain motions, should cease, and the whole system be left to act according to the mechanical laws, the vessel and globe, by means of the intervening fluid, will act upon each other, and will continue to propagate their motions

through the fluid to each other, till their periodic times become equal among themselves, and the whole system revolves together like one solid body.

SCHOLIUM.

In all these reasonings I suppose the fluid to consist of matter of uniform density and fluidity; I mean, that the fluid is such, that a globe placed any where therein may propagate with the same motion of its own, at distances from itself continually equal, similar and equal motions in the fluid in the same interval of time. The matter by its circular motion endeavours to recede from the axis of the vortex, and therefore presses all the matter that lies beyond. This pressure makes the attrition greater, and the separation of the parts more difficult; and by consequence diminishes the fluidity of the matter. Again; if the parts of the fluid are in any one place denser or larger than in the others, the fluidity will be less in that place, because there are fewer superficieses where the parts can be separated from each other. In these cases I suppose the defect of the fluidity to be supplied by the smoothness or softness of the parts, or some other condition; otherwise the matter where it is less fluid will cohere more, and be more sluggish, and therefore will receive the motion more slowly, and propagate it farther than agrees with the ratio above assigned. If the vessel be not spherical, the particles will move in lines not circular, but answering to the figure of the vessel; and the periodic times will be nearly as the squares of the mean distances from the centre. In the parts between the centre and the circumference the motions will be slower where the spaces are wide, and swifter where narrow; but yet the particles will not tend to the circumference at all the more for their greater swiftness; for they then describe arcs of less curvity, and the conatus of receding from the centre is as much diminished by the diminution of this curvature as it is augmented by the increase of the velocity. As they go out of narrow into wide spaces, they recede a little farther from the centre, but in doing so are retarded; and when they come out of wide into narrow spaces, they are again accelerated; and so each particle is retarded and accelerated by turns for

ever. These things will come to pass in a rigid vessel; for the state of vortices in an infinite fluid is known by cor. 6 of this proposition.

I have endeavoured in this proposition to investigate the properties of vortices, that I might find whether the celestial phenomena can be explained by them; for the phenomenon is this, that the periodic times of the planets revolving about Jupiter are in the sesquiplicate ratio of their distances from Jupiter's centre; and the same rule obtains also among the planets that revolve about the sun. And these rules obtain also with the greatest accuracy, as far as has been yet discovered by astronomical observation. Therefore if those planets are carried round in vortices revolving about Jupiter and the sun, the vortices must revolve according to that law. But here we found the periodic times of the parts of the vortex to be in the duplicate ratio of the distances from the centre of motion; and this ratio cannot be diminished and reduced to the sesquiplicate, unless either the matter of the vortex be more fluid the farther it is from the centre, or the resistance arising from the want of lubricity in the parts of the fluid should, as the velocity with which the parts of the fluid are separated goes on increasing, be augmented with it in a greater ratio than that in which the velocity increases. But neither of these suppositions seem reasonable. The more gross and less fluid parts will tend to the circumference, unless they are heavy towards the centre. And though, for the sake of demonstration, I proposed, at the beginning of this section, an hypothesis that the resistance is proportional to the velocity, nevertheless, it is in truth probable that the resistance is in a less ratio than that of the velocity; which granted, the periodic times of the parts of the vortex will be in a greater than the duplicate ratio of the distances from its centre. If, as some think, the vortices move more swiftly near the centre, then slower to a certain limit, then again swifter near the circumference, certainly neither the sesquiplicate, nor any other certain and determinate ratio, can obtain in them. Let philosophers then see how that phenomenon of the sesquiplicate ratio can be accounted for by vortices.

PROPOSITION LIII. THEOREM XLI.

Bodies carried about in a vortex, and returning in the same orb, are of the same density with the vortex, and are moved according to the same law with the parts of the vortex, as to velocity and direction of motion.

For if any small part of the vortex, whose particles or physical points preserve a given situation among each other, be supposed to be congealed, this particle will move according to the same law as before, since no change is made either in its density, *vis insita*, or figure. And again; if a congealed or solid part of the vortex be of the same density with the rest of the vortex, and be resolved into a fluid, this will move according to the same law as before, except in so far as its particles, now become fluid, may be moved among themselves. Neglect, therefore, the motion of the particles among themselves as not at all concerning the progressive motion of the whole, and the motion of the whole will be the same as before. But this motion will be the same with the motion of other parts of the vortex at equal distances from the centre; because the solid, now resolved into a fluid, is become perfectly like to the other parts of the vortex. Therefore a solid, if it be of the same density with the matter of the vortex, will move with the same motion as the parts thereof, being relatively at rest in the matter that surrounds it. If it be more dense, it will endeavour more than before to recede from the centre; and therefore overcoming that force of the vortex, by which, being, as it were, kept in equilibrio, it was retained in its orbit, it will recede from the centre, and in its revolution describe a spiral, returning no longer into the same orbit. And, by the same argument, if it be more rare, it will approach to the centre. Therefore it can never continually go round in the same orbit, unless it be of the same density with the fluid. But we have shewn in that case that it would revolve according to the same law with those parts of the fluid that are at the same or equal distances from the centre of the vortex.

COR. 1. Therefore a solid revolving in a vortex, and continually going round in the same orbit, is relatively quiescent in the fluid that carries it.

COR. 2. And if the vortex be of an uniform density, the same body may revolve at any distance from the centre of the vortex.

SCHOLIUM.

Hence it is manifest that the planets are not carried round in corporeal vortices; for, according to the *Copernican* hypothesis, the planets going round the sun revolve in ellipses, having the sun in their common focus; and by radii drawn to the sun describe areas proportional to the times. But now the parts of a vortex can never revolve with such a motion. Let AD, BE, CF (Pl. 9, Fig 3), represent three orbits described about the sun S, of which let the utmost circle CF be concentric to the sun; and let the aphelia of the two innermost be A, B; and their perihelia D, E. Therefore a body revolving in the orb CF, describing, by a radius drawn to the sun, areas proportional to the times, will move with an uniform motion. And, according to the laws of astronomy, the body revolving in the orb BE will move slower in its aphelion B, and swifter in its perihelion E; whereas, according to the laws of mechanics, the matter of the vortex ought to move more swiftly in the narrow space between A and C than in the wide space between D and F; that is, more swiftly in the aphelion than in the perihelion. Now these two conclusions contradict each other. So at the beginning of the sign of Virgo, where the aphelion of Mars is at present, the distance between the orbits of Mars and Venus is to the distance between the same orbits, at the beginning of the sign of Pisces, as about 3 to 2; and therefore the matter of the vortex between those orbits ought to be swifter at the beginning of Pisces than at the beginning of Virgo in the ratio of 3 to 2; for the narrower the space is through which the same quantity of matter passes in the same time of one revolution, the greater will be the velocity with which it passes through it. Therefore if the earth being relatively at rest in this celestial matter should be carried round by it, and revolve

together with it about the sun, the velocity of the earth at the beginning of Pisces would be to its velocity at the beginning of Virgo in a sesquialteral ratio. Therefore the sun's apparent diurnal motion at the beginning of Virgo ought to be above 70 minutes, and at the beginning of Pisces less than 48 minutes; whereas, on the contrary, that apparent motion of the sun is really greater at the beginning of Pisces than at the beginning of Virgo, as experience testifies; and therefore the earth is swifter at the beginning of Virgo than at the beginning of Pisces; so that the hypothesis of vortices is utterly irreconcilable with astronomical phenomena, and rather serves to perplex than explain the heavenly motions. How these motions are performed in free spaces without vortices, may be understood by the first book; and I shall now more fully treat of it in the following book.

BOOK III.

IN the preceding books I have laid down the principles of philosophy; principles not philosophical, but mathematical; such, to wit, as we may build our reasonings upon in philosophical enquiries. These principles are the laws and conditions of certain motions, and powers or forces, which chiefly have respect to philosophy; but, lest they should have appeared of themselves dry and barren, I have illustrated them here and there with some philosophical scholiums, giving an account of such things as are of more general nature, and which philosophy seems chiefly to be founded on; such as the density and the resistance of bodies, spaces void of all bodies, and the motion of light and sounds. It remains that, from the same principles, I now demonstrate the frame of the System of the World. Upon this subject I had, indeed, composed the third book in a popular method, that it might be read by many; but afterwards, considering that such as had not sufficiently entered into the principles could not easily discern the strength of the consequences, nor lay aside the prejudices to which they had been many years accustomed, therefore, to prevent the disputes which might be raised upon

such accounts, I chose to reduce the substance of this book into the form of propositions (in the mathematical way), which should be read by those only who had first made themselves masters of the principles established in the preceding books : not that I would advise any one to the previous study of every proposition of those books ; for they abound with such as might cost too much time, even to readers of good mathematical learning. It is enough if one carefully reads the definitions, the laws of motions, and the first three sections of the first book. He may then pass on to this book, and consult such of the remaining propositions of the first two books, as the references in this, and his occasions, shall require.

RULES OF REASONING IN PHILOSOPHY.

RULE I.

We are to admit no more causes of natural things than such as are both true and sufficient to explain their appearances.

To this purpose the philosophers say that Nature does nothing in vain, and more is in vain when less will serve ; for Nature is pleased with simplicity, and affects not the pomp of superfluous causes.

RULE II.

Therefore to the same natural effects we must, as far as possible, assign the same causes.

As to respiration in a man and in a beast ; the descent of stones in *Europe* and in *America* ; the light of our culinary fire and of the sun ; the reflection of light in the earth and in the planets.

RULE III.

The qualities of bodies, which admit neither intension nor remission of degrees, and which are found to belong to all bodies within the reach of our experiments, are to be esteemed the universal qualities of all bodies whatsoever.

For since the qualities of bodies are only known to us by experiments, we are to hold for universal all such as universally agree with experiments ; and such as are not liable to diminution can never be quite taken away. We are certainly

not to relinquish the evidence of experiments for the sake of dreams and vain fictions of our own devising; nor are we to recede from the analogy of Nature, which uses to be simple, and always consonant to itself. We no other way know the extension of bodies than by our senses, nor do these reach it in all bodies; but because we perceive extension in all that are sensible, therefore we ascribe it universally to all others also. That abundance of bodies are hard, we learn by experience; and because the hardness of the whole arises from the hardness of the parts, we therefore justly infer the hardness of the undivided particles not only of the bodies we feel but of all others. That all bodies are impenetrable, we gather not from reason, but from sensation. The bodies which we handle we find impenetrable, and thence conclude impenetrability to be an universal property of all bodies whatsoever. That all bodies are moveable, and endowed with certain powers (which we call the *vires inertiae*) of persevering in their motion, or in their rest, we only infer from the like properties observed in the bodies which we have seen. The extension, hardness, impenetrability, mobility, and *vis inertiae* of the whole, result from the extension, hardness, impenetrability, mobility, and *vires inertiae* of the parts; and thence we conclude the least particles of all bodies to be also all extended, and hard, and impenetrable, and moveable, and endowed with their proper *vires inertiae*. And this is the foundation of all philosophy. Moreover, that the divided but contiguous particles of bodies may be separated from one another, is matter of observation; and, in the particles that remain undivided, our minds are able to distinguish yet lesser parts, as is mathematically demonstrated. But whether the parts so distinguished, and not yet divided, may, by the powers of Nature, be actually divided and separated from one another, we cannot certainly determine. Yet, had we the proof of but one experiment that any undivided particle, in breaking a hard and solid body, suffered a division, we might by virtue of this rule conclude that the undivided as well as the divided particles may be divided and actually separated to infinity.

Lastly, if it universally appears, by experiments and astro-

nomical observations, that all bodies about the earth gravitate towards the earth, and that in proportion to the quantity of matter which they severally contain; that the moon likewise, according to the quantity of its matter, gravitates towards the earth; that, on the other hand, our sea gravitates towards the moon; and all the planets mutually one towards another; and the comets in like manner towards the sun; we must, in consequence of this rule, universally allow that all bodies whatsoever are endowed with a principle of mutual gravitation. For the argument from the appearances concludes with more force for the universal gravitation of all bodies than for their impenetrability; of which, among those in the celestial regions, we have no experiments, nor any manner of observation. Not that I affirm gravity to be essential to bodies: by their *vis insita* I mean nothing but their *vis inertiae*. This is immutable. Their gravity is diminished as they recede from the earth.

RULE IV.

In experimental philosophy we are to look upon propositions collected by general induction from phaenomena as accurately or very nearly true, notwithstanding any contrary hypotheses that may be imagined, till such time as other phaenomena occur, by which they may either be made more accurate, or liable to exceptions.

This rule we must follow, that the argument of induction may not be evaded by hypotheses.

PHÆNOMENA, OR APPEARANCES.

PHÆNOMENON I.

That the circumjovial planets, by radii drawn to Jupiter's centre, describe areas proportional to the times of description; and that their periodic times, the fixed stars being at rest, are in the sesquiplicate proportion of their distances from its centre.

This we know from astronomical observations. For the orbits of these planets differ but insensibly from circles concentric to Jupiter; and their motions in those circles are found to be uniform. And all astronomers agree that their periodic times are in the sesquiplicate proportion of the semi-diameters of their orbits; and so it manifestly appears from the following table.

The periodic times of the satellites of Jupiter.

1^d. 18^h. 27' 34". 3^d. 13^h. 13' 42". 7^d. 3^h. 42' 36".
16^d. 16^h. 32' 9".

The distances of the satellites from Jupiter's centre.

From the observations of	1	2	3	4	} semi-diameter of Jupiter.
Borelli.....	5 $\frac{2}{3}$	8 $\frac{2}{3}$	14	24 $\frac{2}{3}$	
Townley by the Microm.....	5,52	8,78	13,47	24,72	
Cassini by the Telescope.....	5	8	13	23	
Cassini by the eclip. of the satel.	5 $\frac{2}{3}$	9	14 $\frac{2}{3}$ 0	25 $\frac{3}{10}$ 10	
From the periodic times	5,667	9,017	14,384	25,299	

Mr. Pound has determined, by the help of excellent micrometers, the diameters of Jupiter and the elongation of its satellites after the following manner. The greatest heliocentric elongation of the fourth satellite from Jupiter's centre was taken with a micrometer in a 15 feet telescope, and at the mean distance of Jupiter from the earth was found about 8' 16". The elongation of the third satellite was taken with a micrometer in a telescope of 123 feet, and at the same distance of Jupiter from the earth was found 4' 42". The greatest elongations of the other satellites, at the same distance of Jupiter from the earth, are found from the periodic times to be 2' 56" 47", and 1' 51" 6".

The diameter of Jupiter taken with the micrometer in a 123 feet telescope several times, and reduced to Jupiter's mean distance from the earth, proved always less than 40", never less than 38", generally 39". This diameter in shorter telescopes is 40", or 41"; for Jupiter's light is a little dilated by the unequal refrangibility of the rays, and this dilatation bears a less ratio to the diameter of Jupiter in the longer and more perfect telescopes than in those which are shorter and less perfect. The times in which two satellites, the first and the third, passed over Jupiter's body, were observed, from the beginning of the ingresses to the beginning of the egresses, and from the complete ingresses to the complete egresses, with the long telescope. And from the transit of the first satellite, the diameter of Jupiter at its mean distance from the earth came forth 37 $\frac{1}{8}$ ", and from the transit of the third 37 $\frac{3}{8}$ ". There was observed also the time in which the shadow of the first satellite passed over Jupiter's body, and thence the diameter of Jupiter at its mean distance from the earth came out about

37". Let us suppose its diameter to be $37\frac{1}{4}"$ very nearly, and then the greatest elongations of the first, second, third, and fourth satellite will be respectively equal to 5,965, 9,494, 15,141, and 26,63 semi-diameters of Jupiter.

PHÆNOMENON II.

That the circumsaturnal planets, by radii drawn to Saturn's centre, describe areas proportional to the times of description; and that their periodic times, the fixed stars being at rest, are in the sesquiplicate proportion of their distances from its centre.

For, as *Cassini* from his own observations has determined, their distances from Saturn's centre and their periodic times are as follow.

The periodic times of the satellites of Saturn.

1^d. 21^h. 18' 27". 2^d. 17^h. 41' 22". 4^d. 12^h. 25' 12". 15^d. 22^h.
41' 14". 79^d. 7^h. 48' 00".

The distances of the satellites from Saturn's centre, in semi-diameters of its ring.

From observations..... $1\frac{19}{20}$. $2\frac{1}{2}$. $3\frac{1}{2}$. 8. 24.

From the periodic times..... 1,93. 2,47. 3,45. 8. 23,35.

The greatest elongation of the fourth satellite from Saturn's centre is commonly determined from the observations to be eight of those semi-diameters very nearly. But the greatest elongation of this satellite from Saturn's centre, when taken with an excellent micrometer in Mr. *Huygens's* telescope of 123 feet, appeared to be eight semi-diameters and $\frac{7}{10}$ of a semi-diameter. And from this observation and the periodic times the distances of the satellites from Saturn's centre in semi-diameters of the ring are 2,1. 2,69. 3,75. 8,7. and 25,35. The diameter of Saturn observed in the same telescope was found to be to the diameter of the ring as 3 to 7; and the diameter of the ring, May 28-29, 1719, was found to be 43"; and thence the diameter of the ring when Saturn is at its mean distance from the earth is 42", and the diameter of Saturn 18". These things appear so in very long and excellent telescopes, because in such telescopes the apparent magnitudes of the heavenly bodies bear a greater proportion to the dilatation of light in the extremities of those bodies than in

shorter telescopes. If we, then, reject all the spurious light, the diameter of Saturn will not amount to more than 16".

PHÆNOMENON III.

That the five primary planets, Mercury, Venus, Mars, Jupiter, and Saturn, with their several orbits, encompass the sun.

That Mercury and Venus revolve about the sun, is evident from their moon-like appearances. When they shine out with a full face, they are, in respect of us, beyond or above the sun; when they appear half full, they are about the same height on one side or other of the sun; when horned, they are below or between us and the sun; and they are sometimes, *when directly under*, seen like spots traversing the sun's disk. That Mars surrounds the sun, is as plain from its full face when near its conjunction with the sun, and from the gibbous figure which it shews in its quadratures. And the same thing is demonstrable of Jupiter and Saturn, from their appearing full in all situations; for the shadows of their satellites that appear sometimes upon their disks make it plain that the light they shine with is not their own, but borrowed from the sun.

PHÆNOMENON IV.

That the fixed stars being at rest, the periodic times of the five primary planets, and (whether of the sun about the earth, or) of the earth about the sun, are in the sesquiplicate proportion of their mean distances from the sun.

This proportion, first observed by *Kepler*, is now received by all astronomers; for the periodic times are the same, and the dimensions of the orbits are the same, whether the sun revolves about the earth, or the earth about the sun. And as to the measures of the periodic times, all astronomers are agreed about them. But for the dimensions of the orbits, *Kepler* and *Bullialdus*, above all others, have determined them from observations with the greatest accuracy; and the mean distances corresponding to the periodic times differ but insensibly from those which they have assigned, and for the most part fall in between them; as we may see from the following table.

The periodic times, with respect to the fixed stars, of the planets and earth, revolving about the sun, in days and decimal parts of a day.

h	m	s	s	s
10759,275.	4332,514.	686,9785.	365,2565.	224,6176.
		s		
		87,9692.		

The mean distances of the planets and of the earth from the sun.

	h	m	s
According to <i>Kepler</i>	951000.	519650.	152350.
.....to <i>Bullialdus</i>	954198.	522520.	152350.
.....to the periodic times..	954006.	520096.	152369.
	s	s	s
According to <i>Kepler</i>	100000.	72400.	38806.
.....to <i>Bullialdus</i>	100000.	72398.	38585.
.....to the periodic times....	100000.	72333.	38710.

As to Mercury and Venus, there can be no doubt about their distances from the sun; for they are determined by the elongations of those planets from the sun; and for the distances of the superior planets, all dispute is cut off by the eclipses of the satellites of Jupiter. For by those eclipses the position of the shadow which Jupiter projects is determined; whence we have the heliocentric longitude of Jupiter. And from its heliocentric and geocentric longitudes compared together, we determine its distance.

PHÆNOMENON V.

That the primary planets, by radii drawn to the earth, describe areas nowise proportional to the times; but that the areas which they describe by radii drawn to the sun are proportional to the times of description.

For to the earth they appear sometimes direct, sometimes stationary, nay, and sometimes retrograde. But from the sun they are always seen direct, and to proceed with a motion nearly uniform, that is to say, a little swifter in the perihelion and a little slower in the aphelion distances, so as to maintain an equality in the description of the areas. This is a noted proposition among astronomers, and particularly demonstra-

ble in Jupiter, from the eclipses of his satellites ; by the help of which eclipses, as we have said, the heliocentric longitudes of that planet, and its distances from the sun, are determined.

PHÆNOMENON VI.

That the moon, by a radius drawn to the earth's centre, describes an area proportional to the time of description.

This we gather from the apparent motion of the moon, compared with its apparent diameter. It is true that the motion of the moon is a little disturbed by the action of the sun ; but in laying down these phænomena, I neglect those small and inconsiderable errors.

PROPOSITIONS.

PROPOSITION I. THEOREM I.

That the forces by which the circumjovial planets are continually drawn off from rectilinear motions, and retained in their proper orbits, tend to Jupiter's centre ; and are reciprocally as the squares of the distances of the places of those planets from that centre.

The former part of this proposition appears from phæn. 1, and prop. 2 or 3, book 1 ; the latter from phæn. 1, and cor. 6, prop. 4, of the same book.

The same thing we are to understand of the planets which encompass Saturn, by phæn. 2.

PROPOSITION II. THEOREM II.

That the forces by which the primary planets are continually drawn off from rectilinear motions, and retained in their proper orbits, tend to the sun ; and are reciprocally as the squares of the distances of the places of those planets from the sun's centre.

The former part of the proposition is manifest from phæn. 5, and prop. 2, book 1 ; the latter from phæn. 4, and cor. 6, prop. 4, of the same book. But this part of the proposition is, with great accuracy, demonstrable from the quiescence of the aphe-
 lion points ; for a very small aberration from the reciprocal duplicate proportion would (by cor. 1, prop. 45, book 1) produce a motion of the apsidæ sensible enough in every single revolution, and in many of them enormously great.

PROPOSITION III. THEOREM III.

That the force by which the moon is retained in its orbit tends to the earth; and is reciprocally as the square of the distance of its place from the earth's centre.

The former part of the proposition is evident from phæn. 6, and prop. 2 or 3, book 1; the latter from the very slow motion of the moon's apogee, which, in every single revolution amounting but to $3^{\circ} 3'$ in *consequentia*, may be neglected. For (by cor. 1, prop. 45, book 1) it appears, that, if the distance of the moon from the earth's centre is to the semi-diameter of the earth as D to 1, the force, from which such a motion will result, is reciprocally as $D^{\frac{2}{243}}$, i. e. reciprocally as the power of D , whose exponent is $\frac{2}{243}$; that is to say, in the proportion of the distance something greater than reciprocally duplicate, but which comes $59\frac{3}{4}$ times nearer to the duplicate than to the triplicate proportion. But in regard that this motion is owing to the action of the sun (as we shall afterwards shew), it is here to be neglected. The action of the sun, attracting the moon from the earth, is nearly as the moon's distance from the earth; and therefore (by what we have shewn in cor. 2, prop. 45, book 1) is to the centripetal force of the moon as 2 to 357,45, or nearly so; that is, as 1 to $178\frac{2}{3}$. And if we neglect so inconsiderable a force of the sun, the remaining force, by which the moon is retained in its orb, will be reciprocally as D^2 . This will yet more fully appear from comparing this force with the force of gravity, as is done in the next proposition.

COR. If we augment the mean centripetal force by which the moon is retained in its orb, first in the proportion of $177\frac{2}{3}$ to $178\frac{2}{3}$, and then in the duplicate proportion of the semi-diameter of the earth to the mean distance of the centres of the moon and earth, we shall have the centripetal force of the moon at the surface of the earth; supposing this force, in descending to the earth's surface, continually to increase in the reciprocal duplicate proportion of the height.

PROPOSITION IV. THEOREM IV.

That the moon gravitates towards the earth, and by the force of gravity is continually drawn off from a rectilinear motion, and retained in its orbit.

The mean distance of the moon from the earth in the syzygies in semi-diameters of the earth, is, according to *Ptolemy* and most astronomers, 59; according to *Vendelin* and *Huygens*, 60; to *Copernicus*, $60\frac{1}{3}$; to *Street*, $60\frac{2}{3}$; and to *Tycho*, $56\frac{1}{2}$. But *Tycho*, and all that follow his tables of refraction, making the refractions of the sun and moon (altogether against the nature of light) to exceed the refractions of the fixed stars, and that by four or five minutes *near the horizon*, did thereby increase the moon's *horizontal* parallax by a like number of minutes, that is, by a twelfth or fifteenth part of the whole parallax. Correct this error, and the distance will become about $60\frac{1}{2}$ semi-diameters of the earth, near to what others have assigned. Let us assume the mean distance of 60 diameters in the syzygies; and suppose one revolution of the moon, in respect of the fixed stars, to be completed in $27^d. 7^h. 43'$, as astronomers have determined; and the circumference of the earth to amount to 123249600 *Paris* feet, as the *French* have found by mensuration. And now, if we imagine the moon, deprived of all motion, to be let go, so as to descend towards the earth with the impulse of all that force by which (by cor. prop. 3) it is retained in its orb, it will, in the space of one minute of time, describe in its fall $15\frac{1}{12}$ *Paris* feet. This we gather by a calculus, founded either upon prop. 36, book 1, or (which comes to the same thing) upon cor. 9, prop. 4, of the same book. For the versed sine of that arc, which the moon, in the space of one minute of time, would, by its mean motion, describe at the distance of 60 semi-diameters of the earth, is nearly $15\frac{1}{12}$ *Paris* feet, or more accurately 15 feet 1 inch and 1 line $\frac{2}{3}$. Wherefore, since that force, in approaching to the earth, increases in the reciprocal duplicate proportion of the distance, and, upon that account, at the surface of the earth, is 60×60 times greater than at the moon, a body in our regions, falling with that force, ought, in the space of one minute of time, to describe

$60 \times 60 \times 15\frac{1}{12}$ *Paris* feet; and, in the space of one second of time, to describe $15\frac{1}{12}$ of those feet; or more accurately 15 feet 1 inch and 1 line $\frac{4}{9}$. And with this very force we actually find that bodies here upon earth do really descend; for a pendulum oscillating seconds in the latitude of *Paris* will be 3 *Paris* feet and 8 lines $\frac{1}{2}$ in length, as Mr. *Huygens* has observed. And the space which a heavy body describes by falling in one second of time is to half the length of this pendulum in the duplicate ratio of the circumference of a circle to its diameter (as Mr. *Huygens* has also shewn), and is therefore 15 *Paris* feet 1 inch 1 line $\frac{7}{9}$. And therefore the force by which the moon is retained in its orbit becomes, at the very surface of the earth, equal to the force of gravity which we observe in heavy bodies there. And therefore (by rule 1 and 2) the force by which the moon is retained in its orbit is that very same force which we commonly call gravity; for, were gravity another force different from that, then bodies descending to the earth with the joint impulse of both forces would fall with a double velocity, and in the space of one second of time would describe $30\frac{1}{2}$ *Paris* feet; altogether against experience.

This calculus is founded on the hypothesis of the earth's standing still; for if both earth and moon move about the sun, and at the same time about their common centre of gravity, the distance of the centres of the moon and earth from one another will be $60\frac{1}{2}$ semi-diameters of the earth; as may be found by a computation from prop. 60, book 1.

SCHOLIUM.

The demonstration of this proposition may be more diffusely explained after the following manner. Suppose several moons to revolve about the earth, as in the system of Jupiter or Saturn; the periodic times of these moons (by the argument of induction) would observe the same law which *Kepler* found to obtain among the planets; and therefore their centripetal forces would be reciprocally as the squares of the distances from the centre of the earth, by prop. 1, of this book. Now if the lowest of these were very small, and were so near the earth as almost to touch the tops of the highest mountains,

the centripetal force thereof, retaining it in its orb, would be very nearly equal to the weights of any *terrestrial* bodies that should be found upon the tops of those mountains, as may be known by the foregoing computation. Therefore if the same little moon should be deserted by its centrifugal force that carries it through its orb, and so be disabled from going on-wards therein, it would descend to the earth, and that with the same velocity as heavy bodies do actually fall with upon the tops of those very mountains; because of the equality of the forces that oblige them both to descend. And if the force by which that lowest moon would descend were different from gravity, and if that moon were to gravitate towards the earth, as we find terrestrial bodies do upon the tops of mountains, it would then descend with twice the velocity, as being impelled by both these forces conspiring together. Therefore since both these forces, that is, the gravity of heavy bodies, and the centripetal forces of the moons, respect the centre of the earth, and are similar and equal between themselves, they will (by rule 1 and 2) have one and the same cause. And therefore the force which retains the moon in its orbit is that very force which we commonly call gravity; because otherwise this little moon at the top of a mountain must either be without gravity, or fall twice as swiftly as heavy bodies use to do.

PROPOSITION V. THEOREM V.

That the circumjovial planets gravitate towards Jupiter; the circumsaturnal towards Saturn; the circumfolar towards the sun; and by the forces of their gravity are drawn off from rectilinear motions, and retained in curvilinear orbits.

For the revolutions of the circumjovial planets about Jupiter, of the circumsaturnal about Saturn, and of Mercury and Venus, and the other circumfolar planets, about the sun, are appearances of the same sort with the revolution of the moon about the earth; and therefore, by rule 2, must be owing to the same sort of causes; especially since it has been demonstrated, that the forces upon which those revolutions depend tend to the centres of Jupiter, of Saturn, and of the

sun; and that those forces, in receding from Jupiter, from Saturn, and from the sun, decrease in the same proportion, and according to the same law, as the force of gravity does in receding from the earth.

COR. 1. There is, therefore, a power of gravity tending to all the planets; for, doubtless, Venus, Mercury, and the rest, are bodies of the same sort with Jupiter and Saturn. And since all attraction (by law 3) is mutual, Jupiter will therefore gravitate towards all his own satellites, Saturn towards his, the earth towards the moon, and the sun towards all the primary planets.

COR. 2. The force of gravity which tends to any one planet is reciprocally as the square of the distance of places from that planet's centre.

COR. 3. All the planets do mutually gravitate towards one another, by cor. 1 and 2. And hence it is that Jupiter and Saturn, when near their conjunction, by their mutual attractions sensibly disturb each other's motions. So the sun disturbs the motions of the moon; and both sun and moon disturb our sea, as we shall hereafter explain.

SCHOLIUM.

The force which retains the celestial bodies in their orbits has been hitherto called centripetal force; but it being now made plain that it can be no other than a gravitating force, we shall hereafter call it gravity. For the cause of that centripetal force which retains the moon in its orbit will extend itself to all the planets, by rule 1, 2, and 4.

PROPOSITION VI. THEOREM VI.

That all bodies gravitate towards every planet; and that the weights of bodies towards any the same planet, at equal distances from the centre of the planet, are proportional to the quantities of matter which they severally contain.

It has been, now of a long time, observed by others, that all sorts of heavy bodies (allowance being made for the inequality of retardation which they suffer from a small power of resistance in the air) descend to the earth from equal heights in equal times; and that equality of times we may distinguish to a great accuracy, by the help of pendulums. I tried the

thing in gold, silver, lead, glass, sand, common salt, wood, water, and wheat. I provided two wooden boxes, round and equal: I filled the one with wood, and suspended an equal weight of gold (as exactly as I could) in the centre of oscillation of the other. The boxes hanging by equal threads of 11 feet made a couple of pendulums perfectly equal in weight and figure, and equally receiving the resistance of the air. And, placing the one by the other, I observed them to play together forwards and backwards, for a long time, with equal vibrations. And therefore the quantity of matter in the gold (by cor. 1 and 6, prop. 24, book 2) was to the quantity of matter in the wood as the action of the motive force (or *vis motrix*) upon all the gold to the action of the same upon all the wood; that is, as the weight of the one to the weight of the other: and the like happened in the other bodies. By these experiments, in bodies of the same weight, I could manifestly have discovered a difference of matter less than the thousandth part of the whole, had any such been. But, without all doubt, the nature of gravity towards the planets is the same as towards the earth. For, should we imagine our terrestrial bodies removed to the orb of the moon, and there, together with the moon, deprived of all motion, to be let go, so as to fall together towards the earth, it is certain, from what we have demonstrated before, that, in equal times, they would describe equal spaces with the moon, and of consequence are to the moon, in quantity of matter, as their weights to its weight. Moreover, since the satellites of Jupiter perform their revolutions in times which observe the sesquiquiplicate proportion of their distances from Jupiter's centre, their accelerative gravities towards Jupiter will be reciprocally as the squares of their distances from Jupiter's centre; that is, equal, at equal distances. And, therefore, these satellites, if supposed to fall towards Jupiter from equal heights, would describe equal spaces in equal times, in like manner as heavy bodies do on our earth. And, by the same argument, if the circumsolar planets were supposed to be let fall at equal distances from the sun, they would, in their descent towards the sun, describe equal spaces in equal times. But forces which

equally accelerate unequal bodies must be as those bodies; that is to say, the weights of the planets *towards the sun* must be as their quantities of matter. Further, that the weights of Jupiter and of his satellites towards the sun are proportional to the several quantities of their matter, appears from the exceedingly regular motions of the satellites (by cor. 3, prop. 65, book 1). For if some of those bodies were more strongly attracted to the sun in proportion to their quantity of matter than others, the motions of the satellites would be disturbed by that inequality of attraction (by cor. 2, prop. 65, book 1). If, at equal distances from the sun, any satellite, in proportion to the quantity of its matter, did gravitate towards the sun with a force greater than Jupiter in proportion to his, according to any given proportion, suppose of d to e ; then the distance between the centres of the sun and of the satellite's orbit would be always greater than the distance between the centres of the sun and of Jupiter nearly in the subduplicate of that proportion; as by some computations I have found. And if the satellite did gravitate towards the sun with a force, lesser in the proportion of e to d , the distance of the centre of the satellite's orb from the sun would be less than the distance of the centre of Jupiter from the sun in the subduplicate of the same proportion. Therefore if, at equal distances from the sun, the accelerative gravity of any satellite towards the sun were greater or less than the accelerative gravity of Jupiter towards the sun but by one $\frac{1}{1000}$ part of the whole gravity, the distance of the centre of the satellite's orbit from the sun would be greater or less than the distance of Jupiter from the sun by one $\frac{1}{2000}$ part of the whole distance; that is, by a fifth part of the distance of the utmost satellite from the centre of Jupiter; an eccentricity of the orbit which would be very sensible. But the orbits of the satellites are concentric to Jupiter, and therefore the accelerative gravities of Jupiter, and of all its satellites towards the sun, are equal among themselves. And by the same argument, the weights of Saturn and of his satellites towards the sun, at equal distances from the sun, are as their several quantities of matter; and the weights of the moon and of the earth towards the sun are

either none, or accurately proportional to the masses of matter which they contain. But some they are, by cor. 1 and 3, prop. 5.

But further; the weights of all the parts of every planet towards any other planet are one to another as the matter in the several parts; for if some parts did gravitate more, others less, than for the quantity of their matter, then the whole planet, according to the sort of parts with which it most abounds, would gravitate more or less than in proportion to the quantity of matter in the whole. Nor is it of any moment whether these parts are external or internal; for if, for example, we should imagine the terrestrial bodies with us to be raised up to the orb of the moon, to be there compared with its body; if the weights of such bodies were to the weights of the external parts of the moon as the quantities of matter in the one and in the other respectively, but to the weights of the internal parts in a greater or less proportion, then likewise the weights of those bodies would be to the weight of the whole moon in a greater or less proportion; against what we have shewed above.

COR. 1. Hence the weights of bodies do not depend upon their forms and textures; for if the weights could be altered with the forms, they would be greater or less, according to the variety of forms, in equal matter; altogether against experience.

COR. 2. Universally, all bodies about the earth gravitate towards the earth; and the weights of all, at equal distances from the earth's centre, are as the quantities of matter which they severally contain. This is the quality of all bodies within the reach of our experiments; and therefore (by rule 3) to be affirmed of all bodies whatsoever. If the *æther*, or any other body, were either altogether void of gravity, or were to gravitate less in proportion to its quantity of matter, then, because (according to *Aristotle*, *Des Cartes*, and others) there is no difference betwixt that and other bodies but in mere form of matter, by a successive change from form to form, it might be changed at last into a body of the same condition with those which gravitate most in proportion to their quantity of matter; and, on the other hand, the heaviest bodies,

acquiring the first form of that body, might by degrees quite lose their gravity. And therefore the weights would depend upon the forms of bodies, and with those forms might be changed: contrary to what was proved in the preceding corollary.

COR. 3. All spaces are not equally full; for if all spaces were equally full, then the specific gravity of the fluid which fills the region of the air, on account of the extreme density of the matter, would fall nothing short of the specific gravity of quicksilver, or gold, or any other the most dense body; and, therefore, neither gold, nor any other body, could descend in air; for bodies do not descend in fluids, unless they are specifically heavier than the fluids. And if the quantity of matter in a given space can, by any rarefaction, be diminished, what should hinder a diminution to infinity?

COR. 4. If all the solid particles of all bodies are of the same density, nor can be rarefied without pores, a void, space, or vacuum, must be granted. By bodies of the same density, I mean those whose *vires inertiae* are in the proportion of their bulks.

COR. 5. The power of gravity is of a different nature from the power of magnetism; for the magnetic attraction is not as the matter attracted. Some bodies are attracted more by the magnet; others less; most bodies not at all. The power of magnetism in one and the same body may be increased and diminished; and is sometimes far stronger, for the quantity of matter, than the power of gravity; and in receding from the magnet decreases not in the duplicate but almost in the triplicate proportion of the distance, as nearly as I could judge from some rude observations.

PROPOSITION VII. THEOREM VII.

That there is a power of gravity tending to all bodies, proportional to the several quantities of matter which they contain.

That all the planets mutually gravitate one towards another, we have proved before; as well as that the force of gravity towards every one of them, considered apart, is reciprocally as the square of the distance of places from the centre of the planet. And thence (by prop. 69, book 1, and its corollaries)

it follows, that the gravity tending towards all the planets is proportional to the matter which they contain.

Moreover, since all the parts of any planet A gravitate towards any other planet B; and the gravity of every part is to the gravity of the whole as the matter of the part to the matter of the whole; and (by law 3) to every action corresponds an equal re-action; therefore the planet B will, on the other hand, gravitate towards all the parts of the planet A; and its gravity towards any one part will be to the gravity towards the whole as the matter of the part to the matter of the whole.

Q.E.D.

COR. 1. Therefore the force of gravity towards any whole planet arises from, and is compounded of, the forces of gravity towards all its parts. Magnetic and electric attractions afford us examples of this; for all attraction towards the whole arises from the attractions towards the several parts. The thing may be easily understood in gravity, if we consider a greater planet, as formed of a number of lesser planets, meeting together in one globe; for *hence it would appear that* the force of the whole must arise from the forces of the component parts. If it is objected, that, according to this law, all bodies with us must mutually gravitate one towards another, whereas no such gravitation any where appears, I answer, that since the gravitation towards these bodies is to the gravitation towards the whole earth as these bodies are to the whole earth, the gravitation towards them must be far less than to fall under the observation of our senses.

COR. 2. The force of gravity towards the several equal particles of any body is reciprocally as the square of the distance of places from the particles; as appears from cor. 3, prop. 74, book 1.

PROPOSITION VIII. THEOREM VIII.

In two spheres mutually gravitating each towards the other, if the matter in places on all sides round about and equidistant from the centres is similar, the weight of either sphere towards the other will be reciprocally as the square of the distance between their centres.

After I had found that the force of gravity towards a whole planet did arise from and was compounded of the forces of gravity towards all its parts, and towards every one part was in the reciprocal proportion of the squares of the distances from the part, I was yet in doubt whether that reciprocal duplicate proportion did accurately hold, or but nearly so, in the total force compounded of so many partial ones; for it might be that the proportion which accurately enough took place in greater distances should be wide of the truth near the surface of the planet, where the distances of the particles are unequal, and their situation dissimilar. But by the help of prop. 75 and 76, book 1, and their corollaries, I was at last satisfied of the truth of the proposition, as it now lies before us.

COR. 1. Hence we may find and compare together the weights of bodies towards different planets; for the weights of bodies revolving in circles about planets are (by cor. 2, prop. 4, book 1) as the diameters of the circles directly, and the squares of their periodic times reciprocally; and their weights at the surfaces of the planets, or at any other distances from their centres, are (by this prop.) greater or less in the reciprocal duplicate proportion of the distances. Thus from the periodic times of Venus, revolving about the sun, in $224^d. 16\frac{3}{4}^h.$; of the utmost circumjovial satellite revolving about Jupiter, in $16^d. 16\frac{8}{15}^h.$; of the Huygenian satellite about Saturn in $15^d. 22\frac{2}{3}^h.$; and of the moon about the earth in $27^d. 7^h. 43'$; compared with the mean distance of Venus from the sun, and with the greatest heliocentric elongations of the utmost circumjovial satellite from Jupiter's centre, $8' 16''$; of the Huygenian satellite from the centre of Saturn, $3' 4''$; and of the moon from the earth, $10' 33''$: by computation I found that the weight of equal bodies, at equal distances from the centres of the sun, of Jupiter, of Saturn, and of the earth, towards the sun, Jupiter, Saturn, and the earth, were one to another, as 1, $\frac{1}{1067}$, $\frac{1}{3021}$, and $\frac{1}{169282}$ respectively. Then because as the distances are increased or diminished, the weights are diminished or increased in a duplicate ratio, the weights of equal bodies towards the sun, Jupiter, Saturn, and

the earth, at the distances 10000, 997, 791, and 109 from their centres, that is, at their very superficies, will be as 10000, 943, 529, and 435 respectively. How much the weights of bodies are at the superficies of the moon, will be shewn hereafter.

COR. 2. Hence likewise we discover the quantity of matter in the several planets; for their quantities of matter are as the forces of gravity at equal distances from their centres; that is, in the sun, Jupiter, Saturn, and the earth, as 1, $\frac{1}{1067}$, $\frac{1}{2021}$, and $\frac{1}{169282}$, respectively. If the parallax of the sun be taken greater or less than $10'' 30'''$, the quantity of matter in the earth must be augmented or diminished in the triplicate of that proportion.

COR. 3. Hence also we find the densities of the planets; for (by prop. 72, book 1) the weights of equal and similar bodies towards similar spheres are, at the surfaces of those spheres, as the diameters of the spheres; and therefore the densities of dissimilar spheres are as those weights applied to the diameters of the spheres. But the true diameters of the sun, Jupiter, Saturn, and the earth, were, one to another, as 10000, 997, 791, and 109; and the weights towards the same as 10000, 943, 529, and 435, respectively; and therefore their densities are as 100, $94\frac{1}{2}$, 67, and 400. The density of the earth, which comes out by this computation, does not depend upon the parallax of the sun, but is determined by the parallax of the moon, and therefore is here truly defined. The sun, therefore, is a little denser than Jupiter, and Jupiter than Saturn, and the earth four times denser than the sun; for the sun, by its great heat, is kept in a sort of a rarefied state. The moon is denser than the earth, as shall appear afterwards.

COR. 4. The smaller the planets are, they are, *cæteris paribus*, of so much the greater density; for so the powers of gravity on their several surfaces come nearer to equality. They are likewise, *cæteris paribus*, of the greater density, as they are nearer to the sun. So Jupiter is more dense than Saturn, and the earth than Jupiter; for if the planets were to be placed at different distances from the sun, that, according to their degrees of density, they might enjoy a greater or less

proportion of the sun's heat. Our water, if it were removed as far as the orb of Saturn, would be converted into ice, and in the orb of Mercury would quickly fly away in vapour; for the light of the sun, to which its heat is proportional, is seven times denser in the orb of Mercury than with us: and by the thermometer I have found that a sevenfold heat of our summer sun will make water boil. Nor are we to doubt that the matter of Mercury is adapted to its heat, and is therefore more dense than the matter of our earth; since, in a denser matter, the operations of Nature require a stronger heat.

PROPOSITION IX. THEOREM IX.

That the force of gravity, considered downwards from the surface of the planets, decreases nearly in the proportion of the distances from their centres.

If the matter of the planet were of an uniform density, this proposition would be accurately true (by prop. 73, book 1). The error, therefore, can be no greater than what may arise from the inequality of the density.

PROPOSITION X. THEOREM X.

That the motions of the planets in the heavens may subsist an exceedingly long time.

In the scholium of prop. 40, book 2, I have shewn that a globe of water frozen into ice, and moving freely in our air, in the time that it would describe the length of its semi-diameter, would lose by the resistance of the air $\frac{1}{4588}$ part of its motion; and the same proportion holds nearly in all globes, how great soever, and moved with whatever velocity. But that our globe of earth is of greater density than it would be if the whole consisted of water only, I thus make out. If the whole consisted of water only, whatever was of less density than water, because of its less specific gravity, would emerge and float above. And upon this account, if a globe of terrestrial matter, covered on all sides with water, was less dense than water, it would emerge somewhere; and, the subsiding water falling back, would be gathered to the opposite side. And such is the condition of our earth, which in a great measure is covered with seas. The earth, if it was not for

its greater density, would emerge from the seas, and, according to its degree of levity, would be raised more or less above their surface, the water of the seas flowing backwards to the opposite side. By the same argument, the spots of the sun, which float upon the lucid matter thereof, are lighter than that matter; and, however the planets have been formed while they were yet in fluid masses, all the heavier matter subsided to the centre. Since, therefore, the common matter of our earth on the surface thereof is about twice as heavy as water, and a little lower, in mines, is found about three, or four, or even five times more heavy, it is probable that the quantity of the whole matter of the earth may be five or six times greater than if it consisted all of water; especially since I have before shewed that the earth is about four times more dense than Jupiter. If, therefore, Jupiter is a little more dense than water, in the space of thirty days, in which that planet describes the length of 459 of its semi-diameters, it would, in a medium of the same density with our air, lose almost a tenth part of its motion. But since the resistance of mediums decreases in proportion to their weight or density, so that water, which is $13\frac{1}{2}$ times lighter than quicksilver, resists less in that proportion; and air, which is 860 times lighter than water, resists less in the same proportion; therefore in the heavens, where the weight of the medium in which the planets move is immensely diminished, the resistance will almost vanish.

It is shewn in the scholium of prop. 22, book 2, that at the height of 200 miles above the earth the air is more rare than it is at the superficies of the earth in the ratio of 30 to 0,0000000000003998, or as 75000000000000 to 1, nearly. And hence the planet Jupiter, revolving in a medium of the same density with that superior air, would not lose by the resistance of the medium the 1000000th part of its motion in 1000000 years. In the spaces near the earth the resistance is produced only by the air, exhalations, and vapours. When these are carefully exhausted by the air pump from under the receiver, heavy bodies fall within the receiver with perfect freedom, and without the least sensible resistance: gold itself, and the lightest down, let fall together, will descend with equal

velocity; and though they fall through a space of four, six, and eight feet, they will come to the bottom at the same time, as appears from experiments. And therefore the celestial regions being perfectly void of air and exhalations, the planets and comets, meeting no sensible resistance in those spaces, will continue their motions through them for an immense tract of time.

HYPOTHESIS I.

That the centre of the system of the world is immovable.

This is acknowledged by all, while some contend that the earth, others that the sun, is fixed in that centre. Let us see what may from hence follow.

PROPOSITION XI. THEOREM XI.

That the common centre of gravity of the earth, the sun, and all the planets, is immovable.

For (by cor. 4 of the laws) that centre either is at rest, or moves uniformly forward in a right line; but if that centre moved, the centre of the world would move also, against the hypothesis.

PROPOSITION XII. THEOREM XII.

That the sun is agitated by a perpetual motion, but never recedes far from the common centre of gravity of all the planets.

For since (by cor. 2, prop. 8) the quantity of matter in the sun is to the quantity of matter in Jupiter as 1067 to 1; and the distance of Jupiter from the sun is to the semi-diameter of the sun in a proportion but a small matter greater, the common centre of gravity of Jupiter and the sun will fall upon a point a little without the surface of the sun. By the same argument, since the quantity of matter in the sun is to the quantity of matter in Saturn as 3021 to 1, and the distance of Saturn from the sun is to the semi-diameter of the sun in a proportion but a small matter less, the common centre of gravity of Saturn and the sun will fall upon a point a little within the surface of the sun. And, pursuing the principles of this computation, we should find that though the earth and all the planets were placed on one side of the sun, the distance of the common centre of gravity of all from the centre of the sun would scarcely amount to one diameter

of the sun. In other cases, the distances of those centres are always less; and, therefore, since that centre of gravity is in perpetual rest, the sun, according to the various positions of the planets, must perpetually be moved every way, but will never recede far from that centre.

COR. Hence the common centre of gravity of the earth, the sun, and all the planets, is to be esteemed the centre of the world; for since the earth, the sun, and all the planets, mutually gravitate one towards another, and are therefore, according to their powers of gravity, in perpetual agitation, as the laws of motion require, it is plain that their moveable centres cannot be taken for the immovable centre of the world. If that body were to be placed in the centre, towards which other bodies gravitate most (according to common opinion), that privilege ought to be allowed to the sun; but since the sun itself is moved, a fixed point is to be chosen from which the centre of the sun recedes least, and from which it would recede yet less, if the body of the sun were denser and greater, and therefore less apt to be moved.

PROPOSITION XIII. THEOREM XIII.

The planets move in ellipses which have their common focus in the centre of the sun; and, by radii drawn to that centre, they describe areas proportional to the times of description.

We have discoursed above of these motions from the phenomena. Now that we know the principles on which they depend, from those principles we deduce the motions of the heavens *à priori*. Because the weights of the planets towards the sun are reciprocally as the squares of their distances from the sun's centre, if the sun was at rest, and the other planets did not mutually act one upon another, their orbits would be ellipses, having the sun in their common focus; and they would describe areas proportional to the times of description, by prop. 1 and 11, and cor. 1, prop. 13, book 1. But the mutual actions of the planets one upon another are so very small, that they may be neglected; and, by prop. 66, book 1, they less disturb the motions of the planets around the sun in motion than if those motions were performed about the sun at rest.

It is true, that the action of Jupiter upon Saturn is not to be neglected; for the force of gravity towards Jupiter is to the force of gravity towards the sun as 1 to 1067; and therefore in the conjunction of Jupiter and Saturn, because the distance of Saturn from Jupiter is to the distance of Saturn from the sun almost as 4 to 9, the gravity of Saturn towards Jupiter will be to the gravity of Saturn towards the sun as 81 to 16×1067 ; or, as 1 to about 211. And hence arises a perturbation of the orb of Saturn in every conjunction of this planet with Jupiter, so sensible, that astronomers are puzzled with it. As the planet is differently situated in these conjunctions, its eccentricity is sometimes augmented, sometimes diminished; its aphelion is sometimes carried forwards, sometimes backwards, and its mean motion is by turns accelerated and retarded; yet the whole error in its motion about the sun, though arising from so great a force, may be almost avoided (except in the mean motion) by placing the lower focus of its orbit in the common centre of gravity of Jupiter and the sun (according to prop. 67, book 1), and therefore that error, when it is greatest, scarcely exceeds two minutes; and the greatest error in the mean motion scarcely exceeds two minutes yearly. But in the conjunction of Jupiter and Saturn, the accelerative forces of gravity of the sun towards Saturn, of Jupiter towards Saturn, and of Jupiter towards the sun, are almost as 16, 81, and $\frac{16 \times 81 \times 3021}{25}$, or 156609; and therefore the difference of the forces of gravity of the sun towards Saturn, and of Jupiter towards Saturn, is to the force of gravity of Jupiter towards the sun as 65 to 156609, or as 1 to 2409. But the greatest power of Saturn to disturb the motion of Jupiter is proportional to this difference; and therefore the perturbation of the orbit of Jupiter is much less than that of Saturn's. The perturbations of the other orbits are yet far less, except that the orbit of the earth is sensibly disturbed by the moon. The common centre of gravity of the earth and moon moves in an ellipsis about the sun in the focus thereof, and, by a radius drawn to the sun, describes areas proportional to the times of description. But the earth

in the mean time by a menstrual motion is revolved about this common centre.

PROPOSITION XIV. THEOREM XIV.

The aphelions and nodes of the orbits of the planets are fixed.

The aphelions are immovable, by prop. 11, book 1; and so are the planes of the orbits, by prop. 1 of the same book. And if the planes are fixed, the nodes must be so too. It is true, that some inequalities may arise from the mutual actions of the planets and comets in their revolutions; but these will be so small, that they may be here passed by.

COR. 1. The fixed stars are immovable, seeing they keep the same position to the aphelions and nodes of the planets.

COR. 2. And since these stars are liable to no sensible parallax from the annual motion of the earth, they can have no force, because of their immense distance, to produce any sensible effect in our system. Not to mention that the fixed stars, every where promiscuously dispersed in the heavens, by their contrary attractions destroy their mutual actions, by prop. 70, book 1.

SCHOLIUM.

Since the planets near the sun (viz. Mercury, Venus, the Earth, and Mars) are so small that they can act but with little force upon each other, therefore their aphelions and nodes must be fixed, excepting in so far as they are disturbed by the actions of Jupiter and Saturn, and other higher bodies. And hence we may find, by the theory of gravity, that their aphelions move a little *in consequentia*, in respect of the fixed stars, and that in the sesquiplicate proportion of their several distances from the sun. So that if the aphelion of Mars, in the space of an hundred years, is carried $33' 20''$ *in consequentia*, in respect of the fixed stars, the aphelions of the Earth, of Venus, and of Mercury, will in an hundred years be carried forwards $17' 40''$, $10' 53''$, and $4' 16''$, respectively. But these motions are so inconsiderable, that we have neglected them in this proposition.

PROPOSITION XV. PROBLEM I.

To find the principal diameters of the orbits of the planets.

They are to be taken in the sub-fesquiplicate proportion of the periodic times, by prop. 15, book 1, and then to be severally augmented in the proportion of the sum of the masses of matter in the sun and each planet to the first of two mean proportionals betwixt that sum and the quantity of matter in the sun, by prop. 60, book 1.

PROPOSITION XVI. PROBLEM II.

To find the eccentricities and aphelions of the planets.

This problem is resolved by prop. 18, book 1.

PROPOSITION XVII. THEOREM XV.

That the diurnal motions of the planets are uniform, and that the libration of the moon arises from its diurnal motion.

The proposition is proved from the first law of motion, and cor. 22, prop. 66, book 1. Jupiter, with respect to the fixed stars, revolves in $9^{\text{h}}. 56'$; Mars in $24^{\text{h}}. 39'$; Venus in about 23^{h} ; the Earth in $23^{\text{h}}. 56'$; the Sun in $25\frac{1}{2}$ days; and the moon in 27 days 7 hours $43'$. These things appear by the phenomena. The spots in the sun's body return to the same situation on the sun's disk, with respect to the earth, in $27\frac{1}{2}$ days; and therefore with respect to the fixed stars the sun revolves in about $25\frac{1}{2}$ days. But because the lunar day, arising from its uniform revolution about its axis, is menstrual, *that is, equal to the time of its periodic revolution in its orb*, therefore the same face of the moon will be always nearly turned to the upper focus of its orb; but, as the situation of that focus requires, will deviate a little to one side and to the other from the earth in the lower focus; and this is the libration in longitude; for the libration in latitude arises from the moon's latitude, and the inclination of its axis to the plane of the ecliptic. This theory of the libration of the moon, Mr. N. Mercator, in his Astronomy, published at the beginning of the year 1676, explained more fully out of the letters I sent him. The utmost satellite of Saturn seems to revolve about its axis with a motion like this of the moon, respecting Saturn continually with the same face; for in its revolution round Saturn, as often as it comes to the eastern part of its orbit, it is

scarcely visible, and generally quite disappears; which is like to be occasioned by some spots in that part of its body, which is then turned toward the earth, as M. *Cassini* has observed. So also the utmost satellite of Jupiter seems to revolve about its axis with a like motion, because in that part of its body which is turned from Jupiter it has a spot, which always appears as if it were in Jupiter's own body, whenever the satellite passes between Jupiter and our eye.

PROPOSITION XVIII. THEOREM XVI.

That the axes of the planets are less than the diameters drawn perpendicular to the axes.

The equal gravitation of the parts on all sides would give a spherical figure to the planets, if it was not for their diurnal revolution in a circle. By that circular motion it comes to pass that the parts receding from the axis endeavour to ascend about the equator; and therefore if the matter is in a fluid state, by its ascent towards the equator it will enlarge the diameters there, and by its descent towards the poles it will shorten the axis. So the diameter of Jupiter (by the concurring observations of astronomers) is found shorter betwixt pole and pole than from east to west. And, by the same argument, if our earth was not higher about the equator than at the poles, the seas would subside about the poles, and, rising towards the equator, would lay all things there under water.

PROPOSITION XIX. PROBLEM III.

To find the proportion of the axis of a planet to the diameters perpendicular thereto.

Our countryman, Mr. *Norwood*, measuring a distance of 905751 feet of *London* measure between *London* and *York*, in 1635, and observing the difference of latitudes to be $2^{\circ} 28'$, determined the measure of one degree to be 367196 feet of *London* measure, that is 57300 *Paris* toises. M. *Picart*, measuring an arc of one degree, and $22' 55''$ of the meridian between *Amiens* and *Malvoisine*, found an arc of one degree to be 57060 *Paris* toises. M. *Cassini*, the father, measured the distance upon the meridian from the town of *Collioure* in *Roussillon* to the Observatory of *Paris*; and his son added the distance from the Observatory to the Citadel of *Dunkirk*.

The whole distance was $486156\frac{1}{2}$ toises, and the difference of the latitudes of *Collioure* and *Dunkirk* was 8 degrees, and $31' 11\frac{1}{2}''$. Hence an arc of one degree appears to be 57061 *Paris* toises. And from these measures we conclude that the circumference of the earth is 123249600, and its semi-diameter 19615800 *Paris* feet, upon the supposition that the earth is of a spherical figure.

In the latitude of *Paris* a heavy body falling in a second of time describes 15 *Paris* feet 1 inch 1 line, as above, that is, 2173 lines $\frac{7}{8}$. The weight of the body is diminished by the weight of the ambient air. Let us suppose the weight lost thereby to be $\frac{1}{11000}$ part of the whole weight; then that heavy body falling *in vacuo* will describe a height of 2174 lines in one second of time.

A body in every sidereal day of $23^h. 56' 4''$, uniformly revolving in a circle at the distance of 19615800 feet from the centre, in one second of time describes an arc of 1433,46 feet; the versed sine of which is 0,05236561 feet, or 7,54064 lines. And therefore the force with which bodies descend in the latitude of *Paris* is to the centrifugal force of bodies in the equator arising from the diurnal motion of the earth as 2174 to 7,54064.

The centrifugal force of bodies in the equator is to the centrifugal force with which bodies recede directly from the earth in the latitude of *Paris* $48^\circ 50' 10''$ in the duplicate proportion of the radius to the cosine of the latitude, that is, as 7,54064 to 3,267. Add this force to the force with which bodies descend by their weight in the latitude of *Paris*, and a body in the latitude of *Paris*, falling by its whole undiminished force of gravity, in the time of one second, will describe 2177,267 lines, or 15 *Paris* feet 1 inch and 5,267 lines. And the total force of gravity in that latitude will be to the centrifugal force of bodies in the equator of the earth as 2177,267 to 7,54064, or as 289 to 1.

Wherefore if APBQ (Pl. 10, Fig. 1) represent the figure of the earth, now no longer spherical, but generated by the rotation of an ellipsis about its lesser axis; and ACQqca a canal full of water, reaching from the pole Qq to the centre Cc,

and thence rising to the equator Aa ; the weight of the water in the leg of the canal $ACca$ will be to the weight of water in the other leg $QCcq$ as 289 to 288, because the centrifugal force arising from the circular motion sustains and takes off one of the 289 parts of the weight (in the one leg), and the weight of 288 in the other sustains the rest. But by computation (from cor. 2, prop. 91, book 1) I find, that if the matter of the earth was all uniform, and without any motion, and its axis PQ were to the diameter AB as 100 to 101, the force of gravity in the place Q towards the earth would be to the force of gravity in the same place Q towards a sphere described about the centre C with the radius PC , or QC , as 126 to 125. And, by the same argument, the force of gravity in the place A towards the spheroid generated by the rotation of the ellipsis $APBQ$ about the axis AB is to the force of gravity in the same place A , towards the sphere described about the centre C with the radius AC , as 125 to 126. But the force of gravity in the place A towards the earth is a mean proportional betwixt the forces of gravity towards that spheroid and this sphere; because the sphere, by having its diameter PQ diminished in the proportion of 101 to 100, is transformed into the figure of the earth; and this figure, by having a third diameter perpendicular to the two diameters AB and PQ diminished in the same proportion, is converted into the said spheroid; and the force of gravity in A , in either case, is diminished nearly in the same proportion. Therefore the force of gravity in A towards the sphere described about the centre C with the radius AC , is to the force of gravity in A towards the earth as 126 to $125\frac{1}{2}$. And the force of gravity in the place Q towards the sphere described about the centre C with the radius QC , is to the force of gravity in the place A towards the sphere described about the centre C , with the radius AC , in the proportion of the diameters (by prop. 72, book 1), that is, as 100 to 101. If, therefore, we compound those three proportions 126 to $125\frac{1}{2}$, 126 to $125\frac{1}{2}$, and 100 to 101, into one, the force of gravity in the place Q towards the earth will be to the force of gravity in the place A towards the earth as $126 \times 126 \times 100$ to $125 \times 125\frac{1}{2} \times 101$; or as 501 to 500.

Now since (by cor. 3, prop. 91, book 1) the force of gravity in either leg of the canal ACca, or QCcq, is as the distance of the places from the centre of the earth, if those legs are conceived to be divided by transverse, parallel, and equidistant surfaces, into parts proportional to the wholes, the weights of any number of parts in the one leg ACca will be to the weights of the same number of parts in the other leg as their magnitudes and the accelerative forces of their gravity conjunctly, that is, as 101 to 100, and 500 to 501, or as 505 to 501. And therefore if the centrifugal force of every part in the leg ACca, arising from the diurnal motion, was to the weight of the same part as 4 to 505, so that from the weight of every part, conceived to be divided into 505 parts, the centrifugal force might take off four of those parts, the weights would remain equal in each leg, and therefore the fluid would rest in an equilibrium. But the centrifugal force of every part is to the weight of the same part as 1 to 289; that is, the centrifugal force, which should be $\frac{4}{505}$ parts of the weight, is only $\frac{1}{289}$ part thereof. And, therefore, I say, by the rule of proportion, that if the centrifugal force $\frac{4}{505}$ make the height of the water in the leg ACca to exceed the height of the water in the leg QCcq by one $\frac{1}{100}$ part of its whole height, the centrifugal force $\frac{1}{289}$ will make the excess of the height in the leg ACca only $\frac{1}{289}$ part of the height of the water in the other leg QCcq; and therefore the diameter of the earth at the equator is to its diameter from pole to pole as 230 to 229. And since the mean semi-diameter of the earth, according to *Picart's* mensuration, is 19615800 *Paris* feet, or 3923,16 miles (reckoning 5000 feet to a mile), the earth will be higher at the equator than at the poles by 85472 feet, or $17\frac{1}{10}$ miles. And its height at the equator will be about 19658600 feet, and at the poles 19573000 feet.

If, the density and periodic time of the diurnal revolution remaining the same, the planet was greater or less than the earth, the proportion of the centrifugal force to that of gravity, and therefore also of the diameter betwixt the poles to the diameter at the equator, would likewise remain the same. But if the diurnal motion was accelerated or retarded in any proportion, the centrifugal force would be augmented or

diminished nearly in the same duplicate proportion; and therefore the difference of the diameters will be increased or diminished in the same duplicate ratio very nearly. And if the density of the planet was augmented or diminished in any proportion, the force of gravity tending towards it would also be augmented or diminished in the same proportion; and the difference of the diameters contrariwise would be diminished in proportion as the force of gravity is augmented, and augmented in proportion as the force of gravity is diminished. Wherefore, since the earth, in respect of the fixed stars, revolves in $23^h. 56'$, but Jupiter in $9^h. 56'$, and the squares of their periodic times are as 29 to 5, and their densities as 400 to $94\frac{1}{2}$, the difference of the diameters of Ju-

piter will be to its lesser diameter as $5 \times \frac{400}{94\frac{1}{2}} \times \frac{229}{2}$ to 1, or

as 1 to $9\frac{1}{3}$, nearly. Therefore the diameter of Jupiter from east to west is to its diameter from pole to pole nearly as $10\frac{1}{3}$ to $9\frac{1}{3}$. Therefore since its greatest diameter is $37''$, its lesser diameter lying between the poles will be $33'' 25'''$. Add thereto about $3''$ for the irregular refraction of light, and the apparent diameters of this planet will become $40''$ and $36'' 25'''$; which are to each other as $11\frac{1}{6}$ to $10\frac{1}{6}$, very nearly. These things are so upon the supposition that the body of Jupiter is uniformly dense. But now if its body be denser towards the plane of the equator than towards the poles, its diameters may be to each other as 12 to 11, or 13 to 12, or perhaps as 14 to 13.

And *Cassini* observed, in the year 1691, that the diameter of Jupiter reaching from east to west is greater by about a fifteenth part than the other diameter. Mr. *Pound* with his 123 feet telescope, and an excellent micrometer, measured the diameters of Jupiter in the year 1719, and found them as follow.

The Times.			Greatest diam.	Lesser diam.	The diam. to each other.	
Day.	Hours.		Parts.	Parts.	As	to
January	28	6	13,40	12,28	12	11
March	6	7	13,12	12,20	$13\frac{3}{4}$	$12\frac{3}{4}$
March	9	7	13,12	12,08	$12\frac{3}{5}$	$11\frac{2}{3}$
April	9	9	12,32	11,48	$14\frac{1}{2}$	$13\frac{1}{2}$

So that the theory agrees with the phænomena; for the planets are more heated by the sun's rays towards their equators, and therefore are a little more condensed by that heat than towards their poles.

Moreover, that there is a diminution of gravity occasioned by the diurnal rotation of the earth, and therefore the earth rises higher there than it does at the poles (supposing that its matter is uniformly dense), will appear by the experiments of pendulums related under the following proposition.

PROPOSITION XX. PROBLEM IV.

To find and compare together the weights of bodies in the different regions of our earth.

Because the weights of the unequal legs of the canal of water ACQqca are equal; and the weights of the parts proportional to the whole legs, and alike situated in them, are one to another as the weights of the wholes, and therefore equal betwixt themselves; the weights of equal parts, and alike situated in the legs, will be reciprocally as the legs, that is, reciprocally as 230 to 229. And the case is the same in all homogeneous equal bodies alike situated in the legs of the canal. Their weights are reciprocally as the legs, that is, reciprocally as the distances of the bodies from the centre of the earth. Therefore if the bodies are situated in the uppermost parts of the canals, or on the surface of the earth, their weights will be one to another reciprocally as their distances from the centre. And, by the same argument, the weights in all other places round the whole surface of the earth are reciprocally as the distances of the places from the centre; and, therefore, in the hypothesis of the earth's being a spheroid, are given in proportion.

Whence arises this theorem, that the increase of weight in passing from the equator to the poles is nearly as the versed sine of double the latitude; or, which comes to the same thing, as the square of the right sine of the latitude; and the arcs of the degrees of latitude in the meridian increase nearly in the same proportion. And, therefore, since the latitude of *Paris* is $48^{\circ} 50'$, that of places under the equator $00^{\circ} 00'$, and that of places under the poles 90° ; and the versed sines of double those arcs are 11334,00000 and 20000, the radius be-

ing 10000 ; and the force of gravity at the pole is to the force of gravity at the equator as 230 to 229 ; and the excess of the force of gravity at the pole to the force of gravity at the equator as 1 to 229 ; the excess of the force of gravity in the latitude of *Paris* will be to the force of gravity at the equator as $1 \times \frac{11334}{20000}$ to 229, or as 5667 to 2290000. And therefore the whole forces of gravity in those places will be one to the other as 2295667 to 2290000. Wherefore since the lengths of pendulums vibrating in equal times are as the forces of gravity, and, in the latitude of *Paris*, the length of a pendulum vibrating seconds is 3 *Paris* feet and $8\frac{1}{2}$ lines, or rather, because of the weight of the air, $8\frac{5}{8}$ lines, the length of a pendulum vibrating in the same time under the equator will be shorter by 1,087 lines. And by a like calculus the following table is made.

Latitude of the place.	Length of the pendulum.		Measure of one degree in the meridian.
Deg.	Feet.	Lines.	Toises.
0	3	7,468	56637
5	3	7,482	56642
10	3	7,526	56659
15	3	7,596	56687
20	3	7,692	56724
25	3	7,812	56769
30	3	7,948	56823
35	3	8,099	56882
40	3	8,261	56945
1	3	8,294	56958
2	3	8,327	56971
3	3	8,361	56984
4	3	8,394	56997
45	3	8,428	57010
6	3	8,461	57022
7	3	8,494	57035
8	3	8,528	57048
9	3	8,561	57061
50	3	8,594	57074
55	3	8,756	57137
60	3	8,907	57196
65	3	9,044	57250
70	3	9,162	57295
75	3	9,258	57332
80	3	9,329	57360
85	3	9,372	57377
90	3	9,387	57382

By this table, therefore, it appears that the inequality of degrees is so small, that the figure of the earth, in geographical matters, may be considered as spherical; especially if the earth be a little denser towards the plane of the equator than towards the poles.

Now several astronomers, sent into remote countries to make astronomical observations, have found that pendulum clocks do accordingly move slower near the equator than in our climates. And, first of all, in the year 1672, M. *Richer* took notice of it in the island of *Cayenne*; for when, in the month of *August*, he was observing the transits of the fixed stars over the meridian, he found his clock to go slower than it ought in respect of the mean motion of the sun at the rate of $2' 28''$ a day. Therefore, fitting up a simple pendulum to vibrate in seconds, which were measured by an excellent clock, he observed the length of that simple pendulum; and this he did over and over every week for ten months together. And upon his return to *France*, comparing the length of that pendulum with the length of the pendulum at *Paris* (which was 3 *Paris* feet and $8\frac{2}{3}$ lines), he found it shorter by $1\frac{1}{4}$ line.

Afterwards, our friend Dr. *Halley*, about the year 1677, arriving at the island of St. *Helena*, found his pendulum clock to go slower there than at *London*, without marking the difference. But he shortened the rod of his clock by more than the $\frac{1}{8}$ of an inch, or $1\frac{1}{2}$ line; and to effect this, because the length of the screw at the lower end of the rod was not sufficient, he interposed a wooden ring betwixt the nut and the ball.

Then, in the year 1682, M. *Varin* and M. *des Hayes* found the length of a simple pendulum vibrating in seconds at the Royal Observatory of *Paris* to be 3 feet and $8\frac{5}{8}$ lines. And by the same method in the island of *Goree*, they found the length of an isochronal pendulum to be 3 feet and $6\frac{3}{8}$ lines, differing from the former by two lines. And in the same year, going to the islands of *Guadaloupe* and *Martinico*, they found that the length of an isochronal pendulum in those islands was 3 feet and $6\frac{1}{2}$ lines.

After this, M. *Couplet*, the son, in the month of *July* 1697, at the Royal Observatory of *Paris*, so fitted his pendulum clock to the mean motion of the sun, that for a considerable time together the clock agreed with the motion of the sun. In *November* following, upon his arrival at *Lisbon*, he found his clock to go slower than before at the rate of 2' 13" in 24 hours. And next *March* coming to *Paraiba*, he found his clock to go slower there than at *Paris*, and at the rate 4' 12" in 24 hours; and he affirms, that the pendulum vibrating in seconds was shorter at *Lisbon* by $2\frac{1}{2}$ lines, and at *Paraiba* by $3\frac{2}{3}$ lines, than at *Paris*. He had done better to have reckoned those differences $1\frac{1}{3}$ and $2\frac{5}{9}$; for these differences correspond to the differences of the times 2' 13" and 4' 12". But this gentleman's observations are so gross, that we cannot confide in them.

In the following years, 1699 and 1700, M. *des Hayes*, making another voyage to *America*, determined that in the islands of *Cayenne* and *Granada* the length of the pendulum vibrating in seconds was a small matter less than 3 feet and $6\frac{1}{2}$ lines; that in the island of *St. Christophers* it was 3 feet and $6\frac{3}{4}$ lines; and in the island of *St. Domingo* 3 feet and 7 lines.

And in the year 1704, P. *Feuillé*, at *Puerto Bello* in *America*, found that the length of the pendulum vibrating in seconds was 3 *Paris* feet and only $5\frac{7}{12}$ lines, that is, almost 3 lines shorter than at *Paris*; but the observation was faulty. For afterwards, going to the island of *Martinico*, he found the length of the isochronal pendulum there 3 *Paris* feet and $5\frac{10}{12}$ lines.

Now the latitude of *Paraiba* is $6^{\circ} 38'$ south; that of *Puerto Bello* $9^{\circ} 33'$ north; and the latitudes of the islands *Cayenne*, *Goree*, *Guadaloupe*, *Martinico*, *Granada*, *St. Christophers*, and *St. Domingo*, are respectively $4^{\circ} 55'$, $14^{\circ} 40'$, $14^{\circ} 00'$, $14^{\circ} 44'$, $12^{\circ} 06'$, $17^{\circ} 19'$, and $19^{\circ} 48'$, north. And the excesses of the length of the pendulum at *Paris* above the lengths of the isochronal pendulums observed in those latitudes are a little greater than by the table of the lengths of the pendulum before computed. And therefore the earth is a little higher

under the equator than by the preceding calculus, and a little denser at the centre than in mines near the surface, unless, perhaps, the heats of the torrid zone have a little extended the length of the pendulums.

For M. *Picart* has observed, that a rod of iron, which in frosty weather in the winter season was one foot long, when heated by fire, was lengthened into one foot and $\frac{1}{4}$ line. Afterwards M. *de la Hire* found that a rod of iron, which in the like winter season was 6 feet long, when exposed to the heat of the summer sun, was extended into 6 feet and $\frac{3}{4}$ line. In the former case the heat was greater than in the latter; but in the latter it was greater than the heat of the external parts of a human body; for metals exposed to the summer sun acquire a very considerable degree of heat. But the rod of a pendulum clock is never exposed to the heat of the summer sun, nor ever acquires a heat equal to that of the external parts of a human body; and, therefore, though the 3 feet rod of a pendulum clock will indeed be a little longer in the summer than in the winter season, yet the difference will scarcely amount to $\frac{1}{4}$ line. Therefore the total difference of the lengths of isochronal pendulums in different climates cannot be ascribed to the difference of heat; nor, indeed, to the mistakes of the *French* astronomers. For although there is not a perfect agreement betwixt their observations, yet the errors are so small that they may be neglected; and in this they all agree, that isochronal pendulums are shorter under the equator than at the Royal Observatory of *Paris*, by a difference not less than $1\frac{1}{4}$ line, nor greater than $2\frac{3}{4}$ lines. By the observations of M. *Richer*, in the island of *Cayenne*, the difference was $1\frac{1}{4}$ line. That difference being corrected by those of M. *des Hayes*, becomes $1\frac{1}{2}$ line or $1\frac{3}{4}$ line. By the less accurate observations of others, the same was made about 2 lines. And this disagreement might arise partly from the errors of the observations, partly from the dissimilitude of the internal parts of the earth and the height of mountains; partly from the different heats of the air.

I take an iron rod of 3 feet long to be shorter by a sixth part of one line in winter time with us here in *England* than in the summer. Because of the great heats under the

equator, subduct this quantity from the difference of one line and a quarter observed by M. *Richer*, and there will remain 1 line $\frac{1}{12}$, which agrees very well with $1\frac{87}{1000}$ line collected, by the theory a little before. M. *Richer* repeated his observations, made in the island of *Cayenne*, every week for ten months together, and compared the lengths of the pendulum which he had there noted in the iron rods with the lengths thereof which he observed in *France*. This diligence and care seem to have been wanting to the other observers. If this gentleman's observations are to be depended on, the earth is higher under the equator than at the poles, and that by an excess of about 17 miles; as appeared above by the theory.

PROPOSITION XXI. THEOREM XVII.

That the equinoctial points go backwards, and that the axis of the earth, by a nutation in every annual revolution, twice vibrates towards the ecliptic, and as often returns to its former position.

The proposition appears from cor. 20, prop. 66, book 1; but that motion of nutation must be very small, and, indeed, scarcely perceptible.

PROPOSITION XXII. THEOREM XVIII.

That all the motions of the moon, and all the inequalities of those motions, follow from the principles which we have laid down.

That the greater planets, while they are carried about the sun, may in the mean time carry other lesser planets, revolving about them; and that those lesser planets must move in ellipses which have their foci in the centres of the greater, appears from prop. 65, book 1. But then their motions will be several ways disturbed by the action of the sun, and they will suffer such inequalities as are observed in our moon. Thus our moon (by cor. 2, 3, 4, and 5, prop. 66, book 1) moves faster, and, by a radius drawn to the earth, describes an area greater for the time, and has its orbit less curved, and therefore approaches nearer to the earth in the syzygies than in the quadratures, excepting in so far as these effects are hindered by the motion of eccentricity; for (by cor. 9, prop. 66, book 1) the eccentricity is greatest when the apogee of the moon

is in the syzygies, and least when the same is in the quadratures; and upon this account the perigeon moon is swifter, and nearer to us, but the apogee moon slower, and farther from us, in the syzygies than in the quadratures. Moreover, the apogee goes forwards, and the nodes backwards; and this is done not with a regular but an unequal motion. For (by cor. 7 and 8, prop. 66, book 1) the apogee goes more swiftly forwards in its syzygies, more slowly backwards in its quadratures; and, by the excess of its progress above its regress, advances yearly *in consequentia*. But, contrariwise, the nodes (by cor. 11, prop. 66, book 1) are quiescent in their syzygies, and go fastest back in their quadratures. Farther, the greatest latitude of the moon (by cor. 10, prop. 66, book 1) is greater in the quadratures of the moon than in its syzygies. And (by cor. 6, prop. 66, book 1) the mean motion of the moon is slower in the perihelion of the earth than in its aphelion. And these are the principal inequalities (of the moon) taken notice of by astronomers.

But there are yet other inequalities not observed by former astronomers, by which the motions of the moon are so disturbed, that to this day we have not been able to bring them under any certain rule. For the velocities or horary motions of the apogee and nodes of the moon, and their equations, as well as the difference betwixt the greatest eccentricity in the syzygies, and the least eccentricity in the quadratures, and that inequality which we call the variation, are (by cor. 14, prop. 66, book 1) in the course of the year augmented and diminished in the triplicate proportion of the sun's apparent diameter. And besides (by cor. 1 and 2, lem. 10, and cor. 16, prop. 66, book 1) the variation is augmented and diminished nearly in the duplicate proportion of the time between the quadratures. But in astronomical calculations this inequality is commonly thrown into and confounded with the equation of the moon's centre.

PROPOSITION XXIII. PROBLEM V.

To derive the unequal motions of the satellites of Jupiter and Saturn from the motions of our moon.

From the motions of our moon we deduce the corresponding motions of the moons or satellites of Jupiter in this man-

ner, by cor. 16, prop. 66, book 1. The mean motion of the nodes of the outmost fatellite of Jupiter is to the mean motion of the nodes of our moon in a proportion compounded of the duplicate proportion of the periodic time of the earth about the sun to the periodic time of Jupiter about the sun, and the simple proportion of the periodic time of the fatellite about Jupiter to the periodic time of our moon about the earth; and, therefore, those nodes, in the space of a hundred years, are carried $8^{\circ} 24'$ backwards, or *in antecedentia*. The mean motions of the nodes of the inner satellites are to the mean motion of the nodes of the outmost as their periodic times to the periodic time of the former, by the same corollary, and are thence given. And the motion of the apsis of every fatellite *in consequentia* is to the motion of its nodes *in antecedentia* as the motion of the apogee of our moon to the motion of its nodes (by the same corollary), and is thence given. But the motions of the apsides thus found must be diminished in the proportion of 5 to 9, or of about 1 to 2, on account of a cause which I cannot here descend to explain. The greatest equations of the nodes, and of the apsides of every fatellite, are to the greatest equations of the nodes, and apogee of our moon respectively, as the motions of the nodes and apsides of the satellites, in the time of one revolution of the former equations, to the motions of the nodes and apogee of our moon, in the time of one revolution of the latter equations. The variation of a fatellite seen from Jupiter is to the variation of our moon in the same proportion as the whole motions of their nodes respectively during the times in which the fatellite and our moon (after parting from) are revolved (again) to the sun, by the same corollary; and therefore in the outmost fatellite the variation does not exceed $5'' 12'''$.

PROPOSITION XXIV. THEOREM XIX.

That the flux and reflux of the sea arise from the actions of the sun and moon.

By cor. 19 and 20, prop. 66, book 1, it appears that the waters of the sea ought twice to rise and twice to fall every day, as well lunar as solar; and that the greatest height of the waters in the open and deep seas ought to follow the ap-

pulse of the luminaries to the meridian of the place by a less interval than 6 hours; as happens in all that eastern tract of the *Atlantic* and *Æthiopic* seas between *France* and the *Cape of Good Hope*; and on the coasts of *Chili* and *Peru* in the *South Sea*; in all which shores the flood falls out about the second, third, or fourth hour, unless where the motion propagated from the deep ocean is by the shallowness of the channels, through which it passes to some particular places, retarded to the fifth, sixth, or seventh hour, and even later. The hours I reckon from the appulse of each luminary to the meridian of the place, as well under as above the horizon; and by the hours of the lunar day I understand the 24th parts of that time which the moon, by its apparent diurnal motion, employs to come about again to the meridian of the place which it left the day before. The force of the sun or moon in raising the sea is greatest in the appulse of the luminary to the meridian of the place; but the force impressed upon the sea at that time continues a little while after the impression, and is afterwards increased by a new though less force still acting upon it. This makes the sea rise higher and higher, till this new force becoming too weak to raise it any more, the sea rises to its greatest height. And this will come to pass, perhaps, in one or two hours, but more frequently near the shores in about three hours, or even more, where the sea is shallow.

The two luminaries excite two motions, which will not appear distinctly, but between them will arise one mixed motion compounded out of both. In the conjunction or opposition of the luminaries their forces will be conjoined, and bring on the greatest flood and ebb. In the quadratures the sun will raise the waters which the moon depresses, and depress the waters which the moon raises, and from the difference of their forces the smallest of all tides will follow. And because (as experience tells us) the force of the moon is greater than that of the sun, the greatest height of the waters will happen about the third lunar hour. Out of the syzygies and quadratures, the greatest tide, which by the single force of the moon ought to fall out at the third lunar hour, and by the single force of the sun at the third solar hour, by the compounded forces of

both must fall out in an intermediate time that approaches nearer to the third hour of the moon than to that of the sun. And, therefore, while the moon is passing from the syzygies to the quadratures, during which time the 3d hour of the sun precedes the 3d hour of the moon, the greatest height of the waters will also precede the 3d hour of the moon, and that, by the greatest interval, a little after the octants of the moon; and, by like intervals, the greatest tide will follow the 3d lunar hour, while the moon is passing from the quadratures to the syzygies. Thus it happens in the open sea; for in the mouths of rivers the greater tides come later to their height.

But the effects of the luminaries depend upon their distances from the earth; for when they are less distant, their effects are greater, and when more distant, their effects are less, and that in the triplicate proportion of their apparent diameter. Therefore it is that the sun, in the winter time, being then in its perigee, has a greater effect, and makes the tides in the syzygies something greater and those in the quadratures something less than in the summer season; and every month the moon, while in the perigee, raises greater tides than at the distance of 15 days before or after, when it is in its apogee. Whence it comes to pass that two highest tides do not follow one another in two immediately succeeding syzygies.

The effect of either luminary doth likewise depend upon its declination or distance from the equator; for if the luminary was placed at the pole, it would constantly attract all the parts of the waters without any intension or remission of its action, and could cause no reciprocation of motion. And, therefore, as the luminaries decline from the equator towards either pole, they will, by degrees, lose their force, and on this account will excite lesser tides in the solstitial than in the equinoctial syzygies. But in the solstitial quadratures they will raise greater tides than in the quadratures about the equinoxes; because the force of the moon, then situated in the equator, most exceeds the force of the sun. Therefore the greatest tides fall out in those syzygies, and the least in those quadratures, which happen about the time of both equinoxes: and the greatest tide in the syzygies is always succeeded by the least tide in

the quadratures, as we find by experience. But, because the sun is less distant from the earth in winter than in summer, it comes to pass that the greatest and least tides more frequently appear before than after the vernal equinox, and more frequently after than before the autumnal.

Moreover, the effects of the luminaries depend upon the latitudes of places. Let ApEP (Pl. 10, Fig. 2) represent the earth covered with deep waters; C its centre; P, p its poles; AE the equator; F any place without the equator; Ff the parallel of the place; Dd the correspondent parallel on the other side of the equator; L the place of the moon three hours before; H the place of the earth directly under it; h the opposite place; K, k the places at 90 degrees distance; CH, Ch, the greatest heights of the sea from the centre of the earth; and CK, ck, its least heights: and if with the axes Hh, Kk, an ellipsis is described, and by the revolution of that ellipsis about its longer axis Hh a spheroid HPKhpK is formed, this spheroid will nearly represent the figure of the sea; and CF, Cf, CD, Cd, will represent the heights of the sea in the places Ff, Dd. But farther; in the said revolution of the ellipsis any point N describes the circle NM cutting the parallels Ff, Dd, in any places RT, and the equator AE in S; CN will represent the height of the sea in all those places R, S, T, situated in this circle. Wherefore, in the diurnal revolution of any place F, the greatest flood will be in F, at the third hour after the appulse of the moon to the meridian above the horizon; and afterwards the greatest ebb in Q, at the third hour after the setting of the moon; and then the greatest flood in f, at the third hour after the appulse of the moon to the meridian under the horizon; and, lastly, the greatest ebb in Q, at the third hour after the rising of the moon; and the latter flood in f will be less than the preceding flood in F. For the whole sea is divided into two hemispherical floods, one in the hemisphere KHk on the north side, the other in the opposite hemisphere Khk, which we may therefore call the northern and the southern floods. These floods, being always opposite the one to the other, come by turns to the meridians of all places, after an interval of 12 lunar hours.

And seeing the northern countries partake more of the northern flood, and the southern countries more of the southern flood, thence arise tides, alternately greater and less in all places without the equator, in which the luminaries rise and set. But the greatest tide will happen when the moon declines towards the vertex of the place, about the third hour after the appulse of the moon to the meridian above the horizon; and when the moon changes its declination *to the other side of the equator*, that which was the greater tide will be changed into a lesser. And the greatest difference of the floods will fall out about the times of the solstices; especially if the ascending node of the moon is about the first of Aries. So it is found by experience that the morning tides in winter exceed those of the evening, and the evening tides in summer exceed those of the morning; at *Plymouth* by the height of one foot, but at *Bristol* by the height of 15 inches, according to the observations of *Coleprests* and *Sturmy*.

But the motions which we have been describing suffer some alteration from that force of reciprocation, which the waters, being once moved, retain a little while *by their vis insita*. Whence it comes to pass that the tides may continue for some time, though the actions of the luminaries should cease. This power of retaining the impressed motion lessens the difference of the alternate tides, and makes those tides which immediately succeed after the syzygies greater, and those which follow next after the quadratures less. And hence it is that the alternate tides at *Plymouth* and *Bristol* do not differ much more one from the other than by the height of a foot or 15 inches, and that the greatest tides of all at those ports are not the first but the third after the syzygies. And, besides, all the motions are retarded in their passage through shallow channels, so that the greatest tides of all, in some straits and mouths of rivers, are the fourth or even the fifth after the syzygies.

Farther, it may happen that the tide may be propagated from the ocean through different channels towards the same port, and may pass quicker through some channels than through others; in which case the same tide, divided into two

or more succeeding one another, may compound new motions of different kinds. Let us suppose two equal tides flowing towards the same port from different places, the one preceding the other by 6 hours; and suppose the first tide to happen at the third hour of the appulse of the moon to the meridian of the port. If the moon at the time of the appulse to the meridian was in the equator, every 6 hours alternately there would arise equal floods, which, meeting with as many equal ebbs, would so balance one the other, that for that day the water would stagnate and remain quiet. If the moon then declined from the equator, the tides in the ocean would be alternately greater and less, as was said; and from thence two greater and two lesser tides would be alternately propagated towards that port. But the two greater floods would make the greatest height of the waters to fall out in the middle time betwixt both; and the greater and lesser floods would make the waters to rise to a mean height in the middle time between them, and in the middle time between the two lesser floods the waters would rise to their least height. Thus in the space of 24 hours the waters would come, not twice, as commonly, but once only to their greatest and once only to their least height; and their greatest height, if the moon declined towards the elevated pole, would happen at the 6th or 30th hour after the appulse of the moon to the meridian; and when the moon changed its declination, this flood would be changed into an ebb. An example of all which Dr. *Halley* has given us, from the observations of seamen in the port of *Batsham*, in the kingdom of *Tunquin*, in the latitude of $20^{\circ} 50'$ north. In that port, on the day which follows after the passage of the moon over the equator, the waters stagnate: when the moon declines to the north, they begin to flow and ebb, not twice, as in other ports, but once only every day; and the flood happens at the setting and the greatest ebb at the rising of the moon. This tide increases with the declination of the moon till the 7th or 8th day; then for the 7 or 8 days following it decreases at the same rate as it had increased before, and ceases when the moon changes its declination, crossing over the equator to the south. After which the flood is immediately changed

into an ebb; and thenceforth the ebb happens at the setting and the flood at the rising of the moon; till the moon, again passing the equator, changes its declination. There are two inlets to this port and the neighbouring channels, one from the seas of *China*, between the continent and the island of *Luconia*; the other from the *Indian* sea, between the continent and the island of *Borneo*. But whether there be really two tides propagated through the said channels, one from the *Indian* sea in the space of 12 hours, and one from the sea of *China* in the space of 6 hours, which therefore happening at the 3d and 9th lunar hours, by being compounded together, produce those motions; or whether there be any other circumstances in the state of those seas, I leave to be determined by observations on the neighbouring shores.

Thus I have explained the causes of the motions of the moon and of the sea. Now it is fit to subjoin something concerning the quantity of those motions.

PROPOSITION XXV. PROBLEM VI.

To find the forces with which the sun disturbs the motions of the moon. (Pl. 10, Fig. 3.)

Let *S* represent the sun, *T* the earth, *P* the moon, *CADB* the moon's orbit. In *SP* take *SK* equal to *ST*; and let *SL* be to *SK* in the duplicate proportion of *SK* to *SP*. Draw *LM* parallel to *PT*; and if *ST* or *SK* is supposed to represent the accelerated force of gravity of the earth towards the sun, *SL* will represent the accelerative force of gravity of the moon towards the sun. But that force is compounded of the parts *SM* and *LM*, of which the force *LM*, and that part of *SM* which is represented by *TM*, disturb the motion of the moon, as we have shewn in prop. 66, book 1; and its corollaries. Forasmuch as the earth and moon are revolved about their common centre of gravity, the motion of the earth about that centre will be also disturbed by the like forces; but we may consider the sums both of the forces and of the motions as in the moon, and represent the sum of the forces by the lines *TM* and *ML*, which are analogous to them both. The force *ML* (in its mean quantity) is to the centripetal force by which the moon may be retained in its orbit revolving

about the earth at rest, at the distance PT, in the duplicate proportion of the periodic time of the moon about the earth to the periodic time of the earth about the sun (by cor. 17, prop. 66, book 1); that is, in the duplicate proportion of $27^d. 7^h. 43'$ to $365^d. 6^h. 9'$; or as 1000 to 178725; or as 1 to $178\frac{2}{3}$. But in the 4th prop. of this book we found, that, if both earth and moon were revolved about their common centre of gravity, the mean distance of the one from the other would be nearly $60\frac{1}{2}$ mean semi-diameters of the earth; and the force by which the moon may be kept revolving in its orbit about the earth in rest at the distance PT of $60\frac{1}{2}$ semi-diameters of the earth, is to the force by which it may be revolved in the same time, at the distance of 60 semi-diameters, as $60\frac{1}{2}$ to 60; and this force is to the force of gravity with us very nearly as 1 to 60×60 . Therefore the mean force ML is to the force of gravity on the surface of our earth as $1 \times 60\frac{1}{2}$ to $60 \times 60 \times 60 \times 178\frac{2}{3}$, or as 1 to 638092,6; whence, by the proportion of the lines TM, ML, the force TM is also given; and these are the forces with which the sun disturbs the motions of the moon. Q.E.I.

PROPOSITION XXVI. PROBLEM VII.

To find the horary increment of the area which the moon, by a radius drawn to the earth, describes in a circular orbit.

We have above shewn that the area which the moon describes by a radius drawn to the earth is proportional to the time of description, excepting in so far as the moon's motion is disturbed by the action of the sun; and here we propose to investigate the inequality of the moment, or horary increment of that area or motion so disturbed. To render the calculus more easy, we shall suppose the orbit of the moon to be circular, and neglect all inequalities but that only which is now under consideration; and, because of the immense distance of the sun, we shall farther suppose that the lines SP and ST are parallel. By this means, the force LM (Pl. 10, Fig. 4) will be always reduced to its mean quantity TP, as well as the force TM to its mean quantity 3PK. These forces (by cor. 2 of the laws of motion) compose the force TL; and this force, by letting fall the perpendicular LE upon the radius

TP, is resolved into the forces TE, EL; of which the force TE, acting constantly in the direction of the radius TP, neither accelerates nor retards the description of the area TPC made by that radius TP; but EL, *acting on the radius* TP in a perpendicular direction, accelerates or retards *the description of the area* in proportion as it accelerates or retards the moon. The acceleration of the moon, in its passage from the quadrature C to the conjunction A, is in every moment of time as the *generating* accelerative force EL, that is, as $\frac{3PK \times TK}{TP}$.

Let the time be represented by the mean motion of the moon, or (which comes to the same thing) by the angle CTP, or even by the arc CP. At right angles upon CT erect CG equal to CT; and, supposing the quadrantal arc AC to be divided into an infinite number of equal parts Pp, &c. these *parts* may represent the like *infinite* number of the equal parts of time. Let fall pk perpendicular on CT, and draw TG meeting with KP, kp, produced in F and f; then will FK be equal to TK, and Kk be to PK as Pp to Tp, that is, in a given proportion; and therefore $FK \times Kk$, or the area FKkf, will be as $\frac{3PK \times TK}{TP}$, that is, as EL; and compounding, the whole

area GCKF will be as the sum of all the forces EL impressed upon the moon in the whole time CP; and therefore also as the velocity generated by that sum, that is, as the acceleration of the description of the area CTP, or as the increment of the moment *thereof*. The force by which the moon may, in its periodic time CADB of $27^d. 7^h. 43'$, be retained revolving about the earth in rest at the distance TP, would cause a body falling in the time CT to describe the length $\frac{1}{2}CT$, and at the same time to acquire a velocity equal to that with which the moon is moved in its orbit. This appears from cor. 9, prop. 4, book 1. But since Kd, drawn perpendicular on TP, is *but* a third part of EL, and *equal* to the half of TP, or ML, in the octants, the force EL in the octants, where it is greatest, will exceed the force ML in the proportion of 3 to 2; and therefore will be to that force by which the moon in its periodic time may be retained revolving about

the earth at rest as 100 to $\frac{2}{3} \times 17872\frac{1}{2}$, or 11915; and in the time CT will generate a velocity equal to $\frac{100}{11915}$ parts of the velocity of the moon; but in the time CPA will generate a greater velocity in the proportion of CA to CT or TP. Let the greatest force EL in the octants be represented by the area FK $\times$ Kk, or by the rectangle $\frac{1}{2}$ TP $\times$ Pp, which is equal thereto; and the velocity which that greatest force can generate in any time CP will be to the velocity which any other lesser force EL can generate in the same time as the rectangle TP $\times$ CP to the area KCGF. But the velocities generated in the whole time CPA will be one to the other as the rectangle $\frac{1}{2}$ TP $\times$ CA to the triangle TCG, or as the quadrantal arc CA to the radius TP; and therefore the latter velocity generated in the whole time will be $\frac{100}{11915}$ parts of the velocity of the moon. To this velocity of the moon, which is proportional to the mean moment of the area (supposing this mean moment to be represented by the number 11915), we add and subtract the half of the other velocity; the sum 11915 + 50, or 11965, will represent the greatest moment of the area in the syzygy A; and the difference 11915 — 50, or 11865, the least moment thereof in the quadratures. Therefore the areas which in equal times are described in the syzygies and quadratures are one to the other as 11965 to 11865. And if to the least moment 11865 we add a moment which shall be to 100, the difference of the two former moments, as the trapezium FKCG to the triangle TCG, or, which comes to the same thing, as the square of the sine PK to the square of the radius TP (that is, as Pd to TP), the sum will represent the moment of the area when the moon is in any intermediate place P.

But these things take place only in the hypothesis that the sun and the earth are at rest, and that the synodical revolution of the moon is finished in 27^d. 7^h. 43'. But since the moon's synodical period is really 29^d. 12^h. 44', the increments of the moments must be enlarged in the same proportion as the time is, that is, in the proportion of 1080853 to 1000000. Upon which account, the whole increment, which was $\frac{100}{11915}$ parts of the mean moment, will now become $\frac{100}{11023}$ parts thereof; and therefore the moment of the area

in the quadrature of the moon will be to the moment thereof in the syzygy as 11023 — 50 to 11023 + 50; or as 10973 to 11073; and to the moment thereof, when the moon is in any intermediate place P, as 10973 to 10973 + Pd; that is, supposing $TP = 100$.

The area, therefore, which the moon, by a radius drawn to the earth, describes in the several little equal parts of time, is nearly as the sum of the number 219,46, and the versed sine of the double distance of the moon from the nearest quadrature, considered in a circle which hath unity for its radius. Thus it is when the variation in the octants is in its mean quantity. But if the variation there is greater or less, that versed sine must be augmented or diminished in the same proportion.

PROPOSITION XXVII. PROBLEM VIII.

From the horary motion of the moon to find its distance from the earth.

The area which the moon, by a radius drawn to the earth, describes in every moment of time, is as the horary motion of the moon and the square of the distance of the moon from the earth conjunctly. And therefore the distance of the moon from the earth is in a proportion compounded of the subduplicate proportion of the area directly, and the subduplicate proportion of the horary motion inversely. Q.E.I.

COR. 1. Hence the apparent diameter of the moon is given; for it is reciprocally as the distance of the moon from the earth. Let astronomers try how accurately this rule agrees with the phenomena.

COR. 2. Hence also the orbit of the moon may be more exactly defined from the phenomena than hitherto could be done.

PROPOSITION XXVIII. PROBLEM IX.

To find the diameters of the orbit, in which, without eccentricity, the moon would move.

The curvature of the orbit which a body describes, if attracted in lines perpendicular to the orbit, is as the force of attraction directly, and the square of the velocity inversely. I estimate the curvatures of lines compared one with another according to the evanescent proportion of the sines or tan-

gents of their angles of contact to equal radii, supposing those radii to be infinitely diminished. But the attraction of the moon towards the earth in the syzygies is the excess of its gravity towards the earth above the force of the sun $2PK$ (see Fig. prop. 25), by which force the accelerative gravity of the moon towards the sun exceeds the accelerative gravity of the earth towards the sun, or is exceeded by it. But in the quadratures that attraction is the sum of the gravity of the moon towards the earth, and the sun's force KT , by which the moon is attracted towards the earth. And these attractions, putting N for $\frac{AT + CT}{2}$, are nearly as $\frac{178725}{AT^2} + \frac{2000}{CT \times N}$ and $\frac{178725}{CT^2} + \frac{1000}{AT \times N}$, or as $178725N \times CT^2 - 2000AT^2 \times CT$, and $178725N \times AT^2 + 1000CT^2 \times AT$. For if the accelerative gravity of the moon towards the earth be represented by the number 178725, the mean force ML , which in the quadratures is PT or TK , and draws the moon towards the earth, will be 1000, and the mean force TM in the syzygies will be 3000; from which, if we subtract the mean force ML , there will remain 2000, the force by which the moon in the syzygies is drawn from the earth, and which we above called $2PK$. But the velocity of the moon in the syzygies A and B is to its velocity in the quadratures C and D as CT to AT , and the moment of the area, which the moon by a radius drawn to the earth describes in the syzygies, to the moment of that area described in the quadratures conjunctly; that is, as 11073 CT to 10973 AT . Take this ratio twice inversely, and the former ratio once directly, and the curvature of the orb of the moon in the syzygies will be to the curvature thereof in the quadratures as $120406729 \times 178725AT^2 \times CT^2 \times N - 120406729 \times 2000AT^4 \times CT$ to $122611329 \times 178725AT^2 \times CT^2 \times N + 122611329 \times 1000CT^4 \times AT$, that is, as $2151969AT \times CT \times N - 24081AT^3$ to $2191371AT \times CT \times N + 12261CT^3$.

Because the figure of the moon's orbit is unknown, let us, in its stead, assume the ellipsis $DBCA$ (Pl. 10, Fig. 5), in the centre of which we suppose the earth to be situated, and the

greater axis de to lie between the quadratures as the lesser AB between the syzygies. But since the plane of this ellipsis is revolved about the earth by an angular motion, and the orbit, whose curvature we now examine, should be described in a plane void of such motion, we are to consider the figure which the moon, while it is revolved in that ellipsis, describes in this plane, that is to say, the figure Cpa, the several points p of which are found by assuming any point P in the ellipsis, which may represent the place of the moon, and drawing Tp equal to TP in such manner that the angle PTP may be equal to the apparent motion of the sun from the time of the last quadrature in C; or (which comes to the same thing) that the angle CTP may be to the angle CTP as the time of the synodic revolution of the moon to the time of the periodic revolution thereof, or as $29^{\circ}. 12'. 44''$ to $27^{\circ}. 7'. 43''$. If, therefore, in this proportion we take the angle CTa to the right angle CTA, and make Ta of equal length with TA, we shall have a the lower and C the upper apsis of this orbit. But, by computation, I find that the difference betwixt the curvature of this orbit Cpa at the vertex a, and the curvature of a circle described about the centre T with the interval TA, is to the difference betwixt the curvature of the ellipsis at the vertex A, and the curvature of the same circle, in the duplicate proportion of the angle CTP to the angle CTP; and that the curvature of the ellipsis in A is to the curvature of that circle in the duplicate proportion of TA to TC; and the curvature of that circle to the curvature of a circle described about the centre T with the interval TC as TC to TA; but that the curvature of this *last arch* is to the curvature of the ellipsis in C in the duplicate proportion of TA to TC; and that the difference betwixt the curvature of the ellipsis in the vertex C, and the curvature of this last circle, is to the difference betwixt the curvature of the figure Tpa, at the vertex C, and the curvature of this same *last circle*, in the duplicate proportion of the angle CTP to the angle CTP; all which proportions are easily drawn from the sines of the angles of contact, and of the differences of those angles. But, by comparing those proportions together, we find the curvature of the figure

Cpa at a to be to its curvature at C as $AT^3 \times \frac{16824}{100000} CT^2 AT$ to $CT^3 + \frac{16824}{100000} AT^2 \times CT$; where the number $\frac{16824}{100000}$ represents the difference of the squares of the angles CTP and CTP, applied to the square of the lesser angle CTP; or (which is all one) the difference of the squares of the times $27^d. 7^h. 43'$, and $29^d. 12^h. 44'$, applied to the square of the time $27^d. 7^h. 43'$.

Since, therefore, a represents the syzygy of the moon, and C its quadrature, the proportion now found must be the same with that proportion of the curvature of the moon's orb in the syzygies to the curvature thereof in the quadratures, which we found above. Therefore, in order to find the proportion of CT to AT, let us multiply the extremes and the means, and the terms which come out, applied to $AT \times CT$, become $2062,79CT^4 - 2151969N \times CT^3 + 368676N \times AT \times CT^2 + 36342AT^2 \times CT^2 - 362047N \times AT^2 \times CT + 2191371N \times AT^3 + 4051,4AT^4 = 0$. Now if for the half sum N of the terms AT and CT we put 1, and x for their half difference, then CT will be $= 1 + x$, and $AT = 1 - x$. And substituting those values in the equation, after resolving thereof, we shall find $x = 0,00719$; and from thence the semi-diameter $CT = 1,00719$, and the semi-diameter $AT = 0,99281$, which numbers are nearly as $70\frac{1}{4}$, and $69\frac{1}{4}$. Therefore the moon's distance from the earth in the syzygies is to its distance in the quadratures (setting aside the consideration of eccentricity) as $69\frac{1}{4}$ to $70\frac{1}{4}$; or, in round numbers, as 69 to 70.

PROPOSITION XXIX. PROBLEM X.

To find the variation of the moon.

This inequality is owing partly to the elliptic figure of the moon's orbit, partly to the inequality of the moments of the area which the moon by a radius drawn to the earth describes. If the moon P revolved in the ellipsis DBCA about the earth quiescent in the centre of the ellipsis, and by the radius TP, drawn to the earth, described the area CTP, proportional to the time of description; and the greatest semi-diameter CT of the ellipsis was to the least TA as 70 to 69; the tangent of the angle CTP would be to the tangent of the angle of the mean motion, computed from the quadrature C, as

the semi-diameter TA of the ellipsis to its semi-diameter TC, or as 69 to 70. But the description of the area CTP, as the moon advances from the quadrature to the fyzygy, ought to be in such manner accelerated, that the moment of the area in the moon's fyzygy may be to the moment thereof in its quadrature as 11073 to 10973; and that the excess of the moment in any intermediate place P above the moment in the quadrature may be as the square of the sine of the angle CTP; which we may effect with accuracy enough, if we diminish the tangent of the angle CTP in the subduplicate proportion of the number 10973 to the number 11073, that is, in proportion of the number 68,6877 to the number 69. Upon which account the tangent of the angle CTP will now be to the tangent of the mean motion as 68,6877 to 70; and the angle CTP in the octants, where the mean motion is 45° , will be found $44^\circ 27' 28''$, which subtracted from 45° , the angle of the mean motion, leaves the greatest variation $32' 32''$. Thus it would be, if the moon, in passing from the quadrature to the fyzygy, described an angle CTA of 90 degrees only. But because of the motion of the earth, by which the sun is apparently transferred *in consequentia*, the moon, before it overtakes the sun, describes an angle CTa, greater than a right angle, in the proportion of the time of the synodic revolution of the moon to the time of its periodic revolution, that is, in the proportion of $29^d. 12^h. 44'$ to $27^d. 7^h. 43'$. Whence it comes to pass that all the angles about the centre T are dilated in the same proportion; and the greatest variation, which otherwise would be *but* $32' 32''$, now augmented in the said proportion, becomes $35' 10''$.

And this is its magnitude in the mean distance of the sun from the earth, neglecting the differences which may arise from the curvature of the *orbis magnus*, and the stronger action of the sun upon the moon when horned and new, than when gibbous and full. In other distances of the sun from the earth, the greatest variation is in a proportion compounded of the duplicate proportion of the time of the synodic revolution of the moon (the time of the year being given) directly, and the triplicate proportion of the distance of the sun

from the earth inverſely. And, therefore, in the apogee of the ſun, the greateſt variation is $33' 14''$, and in its perigee $37' 11''$, if the eccentricity of the ſun is to the tranſverſe ſemi-diameter of the *orbis magnus* as $16\frac{1}{8}$ to 1000.

Hitherto we have inveſtigated the variation in an orb not eccentric, in which, to wit, the moon in its octants is always in its mean diſtance from the earth. If the moon, on account of its eccentricity, is more or leſs removed from the earth than if placed in this orb, the variation may be ſomething greater, or ſomething leſs, than according to this rule. But I leave the exceſs or defect to the determination of aſtronomers from the phænomena.

PROPOSITION XXX. PROBLEM XI.

To find the horary motion of the nodes of the moon in a circular orbit. (Pl. 11, Fig. 1.)

Let S represent the ſun, T the earth, P the moon, NPn the orbit of the moon, Npn the orthographic projection of the orbit upon the plane of the ecliptic; N, n the nodes, nTNm the line of the nodes produced indefinitely; PI, PK perpendiculars upon the lines ST, Qq; Pp a perpendicular upon the plane of the ecliptic; A, B the moon's ſyzygies in the plane of the ecliptic; AZ a perpendicular let fall upon Nn, the line of the nodes; Q, q the quadratures of the moon in the plane of the ecliptic, and pK a perpendicular on the line Qq lying between the quadratures. The force of the ſun to diſturb the motion of the moon (by prop. 25) is twofold, one proportional to the line LM, the other to the line MT, in the ſcheme of that propoſition; and the moon by the former force is drawn towards the earth, by the latter towards the ſun, in a direction parallel to the right line ST joining the earth and the ſun. The former force LM acts in the direction of the plane of the moon's orbit, and therefore makes no change upon the ſituation thereof, and is upon that account to be neglected; the latter force MT, by which the plane of the moon's orbit is diſturbed, is the ſame with the force 3PK or 3IT. And this force (by prop. 25) is to the force by which the moon may, in its periodic time, be uniformly revolved in a circle about the earth at reſt, as 3IT to the radius of the circle

multiplied by the number 178,725, or as IT to the radius thereof multiplied by 59,575. But in this calculus, and all that follows, I consider all the lines drawn from the moon to the sun as parallel to the line which joins the earth and the sun; because what inclination there is, almost as much diminishes all effects in some cases as it augments them in others; and we are now enquiring after the mean motions of the nodes, neglecting such niceties as are of no moment, and would only serve to render the calculus more perplexed.

Now suppose PM to represent an arc which the moon describes in the least moment of time, and ML a little line, the half of which the moon, by the impulse of the said force 3IT, would describe in the same time; and joining PL, MP, let them be produced to m and l, where they cut the plane of the ecliptic, and upon Tm let fall the perpendicular PH. Now, since the right line ML is parallel to the plane of the ecliptic, and therefore can never meet with the right line ml which lies in that plane, and yet both those right lines lie in one common plane LMPml, they will be parallel, and upon that account the triangles LMP, lmp will be similar. And seeing MPm lies in the plane of the orbit, in which the moon did move while in the place P, the point m will fall upon the line Nn, which passes through the nodes N, n, of that orbit. And because the force by which the half of the little line LM is generated, if the whole had been together, and at once impressed in the point P, would have generated that whole line, and caused the moon to move in the arc whose chord is LP; that is to say, would have transferred the moon from the plane MPmT into the plane LPIT; therefore the angular motion of the nodes generated by that force will be equal to the angle mTl. But ml is to mP as ML to MP; and since MP, because of the time given, is also given, ml will be as the rectangle ML $\times$ mP, that is, as the rectangle IT $\times$ mP. And if Tml is a right angle, the angle mTl will be as $\frac{ml}{Tm}$,

and therefore as $\frac{IT \times Pm}{Tm}$, that is (because Tm and mP, TP

and PH are proportional), as $\frac{IT \times PH}{TP}$; and, therefore, because TP is given, as $IT \times PH$. But if the angle Tml or STN is oblique, the angle mTl will be yet less, in proportion of the sine of the angle STN to the radius, or AZ to AT. And therefore the velocity of the nodes is as $IT \times PH \times AZ$, or as the solid content of the fines of the three angles TPI, PTN, and STN.

If these are right angles, as happens when the nodes are in the quadratures, and the moon in the syzygy, the little line ml will be removed to an infinite distance, and the angle mTl will become equal to the angle mPl. But in this case the angle mPl is to the angle PTM, which the moon in the same time by its apparent motion describes about the earth, as 1 to 59,575. For the angle mPl is equal to the angle LPM, that is, to the angle of the moon's deflexion from a rectilinear path; which angle, if the gravity of the moon should have then ceased, the said force of the sun 3IT would by itself have generated in that given time; and the angle PTM is equal to the angle of the moon's deflexion from a rectilinear path; which angle, if the force of the sun 3IT should have then ceased, the force alone by which the moon is retained in its orbit would have generated in the same time. And these forces (as we have above shewn) are the one to the other as 1 to 59,575. Since, therefore, the mean horary motion of the moon (in respect of the fixed stars) is $32' 56'' 27''' 12\frac{1}{2}^{iv}$, the horary motion of the node in this case will be $33'' 10''' 33^{iv} 12^v$. But in other cases the horary motion will be to $33'' 10''' 33^{iv} 12^v$, as the solid content of the fines of the three angles TPI, PTN, and STN (or of the distances of the moon from the quadrature, of the moon from the node, and of the node from the sun) to the cube of the radius. And as often as the sine of any angle is changed from positive to negative, and from negative to positive, so often must the regressive be changed into a progreffive, and the progreffive into a regressive motion. Whence it comes to pass that the nodes are progreffive as often as the moon happens to be placed between either quadrature, and the node nearest to that quadrature. In other

cases they are regressive, and, by the excess of the regresses above the progresss, they are monthly transferred *in antecedentia*.

COR. 1. Hence if from P and M, the extreme points of a least arc PM (Pl. 11, Fig. 2), on the line Qq joining the quadratures we let fall the perpendiculars PK, Mk, and produce the same till they cut the line of the nodes Nn in D and d, the horary motion of the nodes will be as the area MPDd, and the square of the time AZ conjunctly. For let PK, PH, and AZ, be the three said lines, viz. PK the line of the distance of the moon from the quadrature, PH the line of the distance of the moon from the node, and AZ the line of the distance of the node from the sun; and the velocity of the node will be as the solid content of $PK \times PH \times AZ$. But PT is to PK as PM to Kk; and, therefore, because PT and PM are given, Kk will be as PK. Likewise AT is to PD as AZ to PH, and therefore PH is as the rectangle $PD \times AZ$; and, by compounding those proportions, $PK \times PH$ is as the solid content $Kk \times PD \times AZ$, and $PK \times PH \times AZ$ as $Kk \times PD \times AZ^2$; that is, as the area PDdM and AZ^2 conjunctly. Q.E.D.

COR. 2. In any given position of the nodes their mean horary motion is half their horary motion in the moon's syzygies; and therefore is to $16''\ 35'''$ $16^{iv}.$ 36^v . as the square of the line of the distance of the nodes from the syzygies to the square of the radius, or as AZ^2 to AT^2 . For if the moon, by an uniform motion, describes the semicircle QAq, the sum of all the areas PDdM, during the time of the moon's passage from Q to M, will make up the area QMdE, terminating at the tangent QE of the circle; and by the time that the moon has arrived at the point n, that sum will make up the whole area EQAn described by the line PD: but when the moon proceeds from n to q, the line PD will fall without the circle, and will describe the area nqe, terminating at the tangent qe of the circle; which area, because the nodes were before regressive, but are now progressive, must be subducted from the former area, and, being itself equal to the area QEN, will leave the semicircle NQAn. While, therefore, the moon describes a semicircle, the sum of all the

areas PDdM will be the area of that semicircle; and while the moon describes a complete circle, the sum of those areas will be the area of the whole circle. But the area PDdM, when the moon is in the syzygies, is the rectangle of the arc PM into the radius PT; and the sum of all the areas, *every one* equal to this area, in the time that the moon describes a complete circle, is the rectangle of the whole circumference into the radius of the circle; and this rectangle, being double the area of the circle, will be double the quantity of the former sum. If, therefore, the nodes went on with that velocity uniformly continued which they acquire in the moon's syzygies, they would describe a space double of that which they describe in fact; and, therefore, the mean motion, by which, if uniformly continued, they would describe the same space with that which they do in fact describe by an unequal motion, is *but* one half of that motion which they are possessed of in the moon's syzygies. Wherefore since their greatest horary motion, if the nodes are in the quadratures, is $33'' 10''' 33^{iv}$. 12^v . their mean horary motion in this case will be $16'' 35''' 16^{iv}$. 36^v . And seeing the horary motion of the nodes is every where as AZ^2 and the area PDdM conjunctly, and, therefore, in the moon's syzygies, the horary motion of the nodes is as AZ^2 and the area PDdM conjunctly, that is (because the area PDdM described in the syzygies is given), as AZ^2 , therefore the mean motion also will be as AZ^2 ; and, therefore, when the nodes are without the quadratures, this motion will be to $16'' 35''' 16^{iv}$. 36^v . as AZ^2 to AT^2 . Q.E.D.

PROPOSITION XXXI. PROBLEM XII.

To find the horary motion of the nodes of the moon in an elliptic orbit. (Pl. 12, Fig. 1.)

Let Qpmaq represent an ellipsis described with the greater axis Qq, and the lesser axis ab; QAqB a circle circumscribed; T the earth in the common centre of both; S the sun; p the moon moving in this ellipsis; and pm an arc which it describes in the least moment of time; N and n the nodes joined by the line Nn; pK and mk perpendiculars upon the axis Qq, produced both ways till they meet the circle in P and M, and the line of the nodes in D and d. And if the moon, by a

radius drawn to the earth, describes an area proportional to the time of description, the horary motion of the node in the ellipsis will be as the area $pDdm$ and AZ^2 conjunctly.

For let PF touch the circle in P , and produced meet TN in F ; and pf touch the ellipsis in p , and produced meet the same TN in f , and both tangents concur in the axis TQ at Y . And let ML represent the space which the moon, by the impulse of the abovementioned force $3IT$ or $3PK$, would describe with a transverse motion, in the mean time, while revolving in the circle, it describes the arc PM ; and ml denote the space which the moon, revolving in the ellipsis, would describe in the same time by the impulse of the same force $3IT$ or $3PK$; and let LP and lp be produced till they meet the plane of the ecliptic in G and g , and FG and fg be joined, of which FG produced may cut pf , pg , and TQ , in c , e , and R respectively; and fg produced may cut TQ in r . Because the force $3IT$ or $3PK$ in the circle is to the force $3IT$ or $3pK$ in the ellipsis as PK to pK , or as AT to aT , the space ML generated by the former force will be to the space ml generated by the latter, as PK to pK ; that is, because of the similar figures $PYKp$ and $FYRc$, as FR to cR . But (because of the similar triangles PLM , PGF) ML is to FG as PL to PG , that is (on account of the parallels Lk , PK , GR), as pl to pe , that is (because of the similar triangles plm , cpe), as lm to ce ; and inversely as LM is to lm , or as FR is to cR , so is FG to ce . And therefore if fg was to ce as fy to cY , that is, as fr to cR (that is, as fr to FR and FR to cR conjunctly, that is, as FT to FT , and FG to ce conjunctly), because the ratio of FG to ce , expunged on both sides, leaves the ratios fg to FG and FT to FT , fg would be to FG as FT to FT ; and, therefore, the angles which FG and fg would subtend at the earth T would be equal to each other. But these angles (by what we have shewn in the preceding proposition) are the motions of the nodes, while the moon describes in the circle the arc PM , in the ellipsis the arc pm ; and therefore the motions of the nodes in the circle and in the ellipsis would be equal to each other. Thus, I say, it would be, if fg was to ce as fY to cY , that is, if fg was equal to $\frac{ce \times fY}{cY}$. But because of

the similar triangles fgp , cep , fg is to ce as fp to cp ; and therefore fg is equal to $\frac{ce \times fp}{cp}$; and therefore the angle which fg subtends in fact is to the former angle which FG subtends, that is to say, the motion of the nodes in the ellipsis is to the motion of the same in the circle as this fg or $\frac{ce \times fp}{cp}$ to the former fg or $\frac{ce \times fY}{cY}$, that is, as $fp \times cY$ to $fY \times cp$, or as fp to fY , and cY to cp ; that is, if ph parallel to TN meet FP in h , as Fh to FY and FY to FP ; that is, as Fh to FP or Dp to DP , and therefore as the area $Dpmd$ to the area $DPMd$. And, therefore, seeing (by corol. 1, prop. 30) the latter area and AZ^2 conjunctly are proportional to the horary motion of the nodes in the circle, the former area and AZ^2 conjunctly will be proportional to the horary motion of the nodes in the ellipsis. Q.E.D.

COR. Since, therefore, in any given position of the nodes, the sum of all the areas $pDdm$, in the time while the moon is carried from the quadrature to any place m , is the area $mpQEd$ terminated at the tangent of the ellipsis QE ; and the sum of all those areas, in one entire revolution, is the area of the whole ellipsis; the mean motion of the nodes in the ellipsis will be to the mean motion of the nodes in the circle as the ellipsis to the circle; that is, as Ta to TA , or 69 to 70. And, therefore, since (by corol. 2, prop. 30) the mean horary motion of the nodes in the circle is to $16'' 35''' 16^{iv}. 36^v$. as AZ^2 to AT^2 , if we take the angle $16'' 21''' 3^{iv}. 30^v$. to the angle $16'' 35''' 16^{iv}. 36^v$. as 69 to 70, the mean horary motion of the nodes in the ellipsis will be to $16'' 21''' 3^{iv}. 30^v$. as AZ^2 to AT^2 ; that is, as the square of the sine of the distance of the node from the sun to the square of the radius.

But the moon, by a radius drawn to the earth, describes the area in the syzygies with a greater velocity than it does that in the quadratures, and upon that account the time is contracted in the syzygies, and prolonged in the quadratures; and together with the time the motion of the nodes is likewise augmented or diminished. But the moment of the area in the quadrature of the moon was to the moment thereof in

the syzygies as 10973 to 11073 ; and therefore the mean moment in the octants is to the excess in the syzygies, and to the defect in the quadratures, as 11023, the half sum of those numbers, to their half difference 50. Wherefore since the time of the moon's mora in the several little equal parts of its orbit is reciprocally as its velocity, the mean time in the octants will be to the excess of the time in the quadratures, and to the defect of the time in the syzygies arising from this cause, nearly as 11023 to 50. But, reckoning from the quadratures to the syzygies, I find that the excess of the moments of the area, in the several places above the least moment in the quadratures, is nearly as the square of the sine of the moon's distance from the quadratures ; and therefore the difference betwixt the moment in any place, and the mean moment in the octants, is as the difference betwixt the square of the sine of the moon's distance from the quadratures, and the square of the sine of 45 degrees, or half the square of the radius ; and the increment of the time in the several places between the octants and quadratures, and the decrement thereof between the octants and syzygies, is in the same proportion. But the motion of the nodes, while the moon describes the several little equal parts of its orbit, is accelerated or retarded in the duplicate proportion of the time ; for that motion, while the moon describes PM, is (*cæteris paribus*) as ML, and ML is in the duplicate proportion of the time. Wherefore the motion of the nodes in the syzygies, in the time while the moon describes given little parts of its orbit, is diminished in the duplicate proportion of the number 11073 to the number 11023 ; and the decrement is to the remaining motion as 100 to 10973, but to the whole motion as 100 to 11073 nearly. But the decrement in the places between the octants and syzygies, and the increment in the places between the octants and quadratures, is to this decrement nearly as the whole motion in these places to the whole motion in the syzygies, and the difference betwixt the square of the sine of the moon's distance from the quadrature, and the half square of the radius, to the half square of the radius conjunctly. Wherefore, if the nodes are in the quadratures, and we take two places, one on one side, one on the other, equally distant

from the octant and other two distant by the same interval, one from the syzygy, the other from the quadrature, and from the decrements of the motions in the two places between the syzygy and octant we subtract the increments of the motions in the two other places between the octant and the quadrature, the remaining decrement will be equal to the decrement in the syzygy, as will easily appear by computation; and therefore the mean decrement, which ought to be subtracted from the mean motion of the nodes, is the fourth part of the decrement in the syzygy. The whole horary motion of the nodes in the syzygies (when the moon by a radius drawn to the earth was supposed to describe an area proportional to the time) was $32'' 42''' 7^{iv}$. And we have shewn that the decrement of the motion of the nodes, in the time while the moon, now moving with greater velocity, describes the same space, was to this motion as 100 to 11073; and therefore this decrement is $17''' 43^{iv} 11^v$. The fourth part of which $4''' 25^{iv} 48^v$. subtracted from the mean horary motion above found, $16'' 21''' 3^{iv} 30^v$., leaves $16'' 16''' 37^{iv} 42^v$. their correct mean horary motion.

If the nodes are without the quadratures, and two places are considered, one on one side, one on the other, equally distant from the syzygies, the sum of the motions of the nodes, when the moon is in those places, will be to the sum of their motions, when the moon is in the same places and the nodes in the quadratures, as AZ^2 to AT^2 . And the decrements of the motions arising from the causes but now explained will be mutually as the motions themselves, and therefore the remaining motions will be mutually betwixt themselves as AZ^2 to AT^2 ; and the mean motions will be as the remaining motions. And, therefore, in any given position of the nodes, their correct mean horary motion is to $16'' 16''' 37^{iv} 42^v$. as AZ^2 to AT^2 ; that is, as the square of the sine of the distance of the nodes from the syzygies to the square of the radius.

PROPOSITION XXXIII. PROBLEM XIII.

To find the mean motion of the nodes of the moon. (Pl. 12, Fig. 2.)

The yearly mean motion is the sum of all the mean horary motions throughout the course of the year. Suppose that the

node is in N, and that, after every hour is elapsed, it is drawn back again to its former place; so that, notwithstanding its proper motion, it may constantly remain in the same situation with respect to the fixed stars; while in the mean time the sun S, by the motion of the earth, is seen to leave the node, and to proceed till it completes its apparent annual course by an uniform motion. Let Aa represent a given least arc, which the right line TS always drawn to the sun, by its intersection with the circle NAn, describes in the least given moment of time; and the mean horary motion (from what we have above shewn) will be as AZ^2 , that is (because AZ and ZY are proportional), as the rectangle of AZ into ZY, that is, as the area AZYa; and the sum of all the mean horary motions from the beginning will be as the sum of all the areas aYZA, that is, as the area NAZ. But the greatest AZYa is equal to the rectangle of the arc Aa into the radius of the circle; and therefore the sum of all these rectangles in the whole circle will be to the like sum of all the greatest rectangles as the area of the whole circle to the rectangle of the whole circumference into the radius, that is, as 1 to 2. But the horary motion corresponding to that greatest rectangle was $16'' 16''' 37^{iv}. 42^v$. and this motion in the complete course of the sidereal year, $365^d. 6^h. 9'$, amounts to $39^\circ 38' 7'' 50'''$, and therefore the half thereof, $19^\circ 49' 3'' 55'''$, is the mean motion of the nodes corresponding to the whole circle. And the motion of the nodes, in the time while the sun is carried from N to A, is to $19^\circ 49' 3'' 55'''$ as the area NAZ to the whole circle.

Thus it would be if the node was after every hour drawn back again to its former place, that so, after a complete revolution, the sun at the year's end would be found again in the same node which it had left when the year begun. But, because of the motion of the node in the mean time, the sun must needs meet the node sooner; and now it remains that we compute the abbreviation of the time. Since, then, the sun, in the course of the year, travels 360 degrees, and the node in the same time by its greatest motion would be carried $39^\circ 38' 7'' 50'''$, or 39,6355 degrees; and the mean motion of the node

in any place N is to its mean motion in its quadratures as AZ^2 to AT^2 ; the motion of the fun will be to the motion of the node in N as $360AT^2$ to $39,6355AZ^2$; that is, as $9,0827646AT^2$ to AZ^2 . Wherefore if we suppose the circumference NAN of the whole circle to be divided into little equal parts, such as Aa, the time in which the fun would describe the little arc Aa, if the circle was quiescent, will be to the time of which it would describe the same arc, supposing the circle together with the nodes to be revolved about the centre T, reciprocally as $9,0827646AT^2$ to $9,0827646AT^2 + AZ^2$; for the time is reciprocally as the velocity with which the little arc is described, and this velocity is the sum of the velocities of both fun and node. If, therefore, the sector NTA represent the time in which the fun by itself, without the motion of the node, would describe the arc NA, and the indefinitely small part ATa of the sector represent the little moment of the time in which it would describe the least arc Aa; and (letting fall aY perpendicular upon Nn) if in AZ we take dZ of such length that the rectangle of dZ into ZY may be to the least part ATa of the sector as AZ^2 to $9,0827646AT^2 + AZ^2$, that is to say, that dZ may be to $\frac{1}{2}AZ$ as AT^2 to $9,0827646AT^2 + AZ^2$; the rectangle of dZ into ZY will represent the decrement of the time arising from the motion of the node, while the arc Aa is described; and if the curve NdGn is the locus where the point d is always found, the curvilinear area NdZ will be as the whole decrement of time while the whole arc NA is described; and, therefore, the excess of the sector NAT above the area NdZ will be as the whole time. But because the motion of the node in a less time is less in proportion of the time, the area AaYZ must also be diminished in the same proportion; which may be done by taking in AZ the line eZ of such length, that it may be to the length of AZ as AZ^2 to $9,0827646AT^2 + AZ^2$; for so the rectangle of eZ into ZY will be to the area AZYa as the decrement of the time in which the arc Aa is described to the whole time in which it would have been described, if the node had been quiescent; and, therefore, that rectangle will be as the decrement of the motion of the node. And if the curve NeFn is the locus of

the point *e*, the whole area NeZ, which is the sum of all the decrements of *that motion*, will be as the whole decrement thereof during the time in which the arc AN is described; and the remaining area NAe will be as the remaining motion, which is the true motion of the node, during the time in which the whole arc NA is described by the joint motions of both sun and node. Now the area of the semicircle is to the area of the figure NeFn found by the method of infinite series nearly as 793 to 60. But the motion corresponding or proportional to the whole circle was $19^{\circ} 49' 3'' 55'''$; and therefore the motion corresponding to double the figure NeFn is $1^{\circ} 29' 58'' 2'''$, which taken from the former motion leaves $18^{\circ} 19' 5'' 53'''$, the whole motion of the node with respect to the fixed stars in the interval between two of its conjunctions with the sun; and this motion subducted from the annual motion of the sun 360° , leaves $341^{\circ} 40' 54'' 7'''$, the motion of the sun in the interval between the same conjunctions. But as this motion is to the annual motion 360° , so is the motion of the node but just now found $18^{\circ} 19' 5'' 53'''$ to its annual motion, which will therefore be $19^{\circ} 18' 1'' 23'''$; and this is the mean motion of the nodes in the sidereal year. By astronomical tables, it is $19^{\circ} 21' 50'''$. The difference is less than $\frac{1}{300}$ part of the whole motion, and seems to arise from the eccentricity of the moon's orbit, and its inclination to the plane of the ecliptic. By the eccentricity of this orbit the motion of the nodes is too much accelerated; and, on the other hand, by the inclination of the orbit, the motion of the nodes is something retarded, and reduced to its just velocity.

PROPOSITION XXXIII. PROBLEM XIV.

To find the true motion of the nodes of the moon. (Pl. 12, Fig. 3.)

In the time which is as the area NTA — NdZ (in the preceding Fig.) that motion is as the area NAe, and is thence given; but because the calculus is too difficult, it will be better to use the following construction of the problem. About the centre C, with any interval CD, describe the circle BEFD; produce DC to A so as AB may be to AC as the mean motion to half the mean true motion when the nodes

are in their quadratures (that is, as $19^{\circ} 18' 1'' 23'''$ to $19^{\circ} 49' 3'' 55'''$; and therefore BC to AC as the difference of those motions $0^{\circ} 31' 2'' 32'''$ to the latter motion $19^{\circ} 49' 3'' 55'''$, that is, as 1 to $38\frac{3}{10}$). Then through the point D draw the indefinite line Gg, touching the circle in D; and if we take the angle BCE, or BCF, equal to the double distance of the sun from the place of the node, as found by the mean motion, and drawing AE or AF cutting the perpendicular DG in G, we take another angle which shall be to the whole motion of the node in the interval between its syzygies (that is, to $9^{\circ} 11' 3''$) as the tangent DG to the whole circumference of the circle BED, and add this *last* angle (for which the angle DAG may be used) to the mean motion of the nodes, while they are passing from the quadratures to the syzygies, and subtract it from their mean motion while they are passing from the syzygies to the quadratures, we shall have their true motion; for the true motion so found will nearly agree with the true motion which comes out from assuming the times as the area NTA — NdZ, and the motion of the node as the area NAe; as whoever will please to examine and make the computations will find: and this is the semi-menstrual equation of the motion of the nodes. But there is also a menstrual equation, but which is by no means necessary for finding of the moon's latitude; for since the variation of the inclination of the moon's orbit to the plane of the ecliptic is liable to a twofold inequality, the one semi-menstrual, the other menstrual, the menstrual inequality of *this variation*, and the menstrual equation of the nodes, so moderate and correct each other, that in computing the latitude of the moon both may be neglected.

Cor. From this and the preceding prop. it appears that the nodes are quiescent in their syzygies, but regressive in their quadratures, by an hourly motion of $16'' 19''' 26^{iv.}$; and that the equation of the motion of the nodes in the octants is $1^{\circ} 30'$; all which exactly agree with the phænomena of the heavens.

SCHOLIUM.

Mr. Machin, Astron., Prof. Gresh., and Dr. Henry Pemberton, separately found out the motion of the nodes by a different

method. Mention has been made of this method in another place. Their several papers, both of which I have seen, contained two propositions, and exactly agreed with each other in both of them. Mr. *Machin's* paper coming first to my hands, I shall here insert it.

OF THE MOTION OF THE MOON'S NODES.

" PROPOSITION I.

" *The mean motion of the sun from the node is defined by a*
 " *geometric mean proportional between the mean motion of*
 " *the sun and that mean motion with which the sun recedes*
 " *with the greatest swiftneſs from the node in the qua-*
 " *dratures.*

" Let T (Pl. 13, Fig. 1) be the earth's place, Nn the line of
 " the moon's nodes at any given time, KTM a perpendicular
 " thereto, TA a right line revolving about the centre with the
 " ſame angular velocity with which the ſun and the node
 " recede from one another, in ſuch ſort that the angle be-
 " tween the quieſcent right line Nn and the revolving line
 " TA may be always equal to the diſtance of the places of
 " the ſun and node. Now if any right line TK be divided into
 " parts TS and SK, and thoſe parts be taken as the mean
 " horary motion of the ſun to the mean horary motion of the
 " node in the quadratures, and there be taken the right line
 " TH, a mean proportional between the part TS and the
 " whole TK, this right line will be proportional to the ſun's
 " mean motion from the node.

" For let there be deſcribed the circle NKnM from the
 " centre T and with the radius TK, and about the ſame cen-
 " tre, with the ſemi-axes TH and TN, let there be deſcribed
 " an ellipſis NHnL; and in the time in which the ſun recedes
 " from the node through the arc Na, if there be drawn the
 " right line Tba, the area of the ſector NTa will be the expo-
 " nent of the ſum of the motions of the ſun and node in the
 " ſame time. Let, therefore, the extremely ſmall arc aA be
 " that which the right line Tba, revolving according to the
 " aboveſaid law, will uniformly deſcribe in a given particle
 " of time, and the extremely ſmall ſector TAa will be as the
 " ſum of the velocities with which the ſun and node are car-

" ried two different ways in that time. Now the sun's velocity
 " is almost uniform, its inequality being so small as scarcely
 " to produce the least inequality in the mean motion of the
 " nodes. The other part of this sum, namely, the mean
 " quantity of the velocity of the node, is increased in the re-
 " cesses from the syzygies in a duplicate ratio of the sine of its
 " distance from the sun (by cor. prop. 31, of this book), and,
 " being greatest in its quadratures with the sun in K, is in the
 " same ratio to the sun's velocity as SK to TS, that is, as (the
 " difference of the squares of TK and TH, or) the rectangle
 " KHM to TH². But the ellipsis NBH divides the sector
 " ATa, the exponent of the sums of these two velocities, into
 " two parts ABba and BTb, proportional to the velocities.
 " For produce BT to the circle in β , and from the point B let
 " fall upon the greater axis the perpendicular BG, which be-
 " ing produced both ways may meet the circle in the points F
 " and f; and because the space ABba is to the sector TBb as
 " the rectangle AB β to BT² (that rectangle being equal to the
 " difference of the squares of TA and TB, because the right
 " line A β is equally cut in T, and unequally in B), therefore
 " when the space ABba is the greatest of all in K, this ratio
 " will be the same as the ratio of the rectangle KHM to HT².
 " But the greatest mean velocity of the node was shewn above
 " to be in that very ratio to the velocity of the sun; and
 " therefore in the quadratures the sector ATa is divided into
 " parts proportional to the velocities. And because the rect-
 " angle KHM is to HT² as FBf to BG², and the rectangle
 " AB β is equal to the rectangle FBf, therefore the little area
 " ABba, where it is greatest, is to the remaining sector TBb
 " as the rectangle AB β to BG². But the ratio of these little
 " areas always was as the rectangle AB β to BT²; and there-
 " fore the little area ABba in the place A is less than its
 " correspondent little area in the quadratures in the duplicate
 " ratio of BG to BT, that is, in the duplicate ratio of the sine
 " of the sun's distance from the node. And therefore the
 " sum of all the little areas ABba, to wit, the space ABN,
 " will be as the motion of the node in the time in which the
 " sun hath been going over the arc NA since he left the node;

“ and the remaining space, namely, the elliptic sector NTB,
 “ will be as the sun’s mean motion in the same time. And
 “ because the mean annual motion of the node is that mo-
 “ tion which it performs in the time that the sun completes
 “ one period of its course, the mean motion of the node from
 “ the sun will be to the mean motion of the sun itself as the
 “ area of the circle to the area of the ellipsis; that is, as the
 “ right line TK to the right line TH, which is a mean pro-
 “ portional between TK and TS; or, which comes to the
 “ same, as the mean proportional TH to the right line TS.

“ PROPOSITION II.

“ *The mean motion of the moon’s nodes being given, to find*
 “ *their true motion.*

“ Let the angle A be the distance of the sun from the
 “ mean place of the node, or the sun’s mean motion from the
 “ node. Then if we take the angle B, whose tangent is to
 “ the tangent of the angle A as TH to TK, that is, in the sub-
 “ duplicate ratio of the mean horary motion of the sun to the
 “ mean horary motion of the sun from the node, when the
 “ node is in the quadrature, that angle B will be the distance
 “ of the sun from the node’s true place. For join FT, and,
 “ by the demonstration of the last proportion, the angle FTN
 “ will be the distance of the sun from the mean place of the
 “ node, and the angle ATN the distance from the true place,
 “ and the tangents of these angles are between themselves as
 “ TK to TH.

“ COR. Hence the angle FTA is the equation of the moon’s
 “ nodes; and the sine of this angle, where it is greatest in the
 “ octants, is to the radius as KH to TK + TH. But the sine
 “ of this equation in any other place A is to the greatest sine
 “ as the sine of the sums of the angles FTN + ATN to the
 “ radius; that is, nearly as the sine of double the distance of
 “ the sun from the mean place of the node (namely, 2FTN)
 “ to the radius.

“ SCHOLIUM.

“ If the mean horary motion of the nodes in the quadra-
 “ tures be $16'' 16''' 37^{iv}. 42^v$. that is, in a whole sidereal year,
 “ $39^\circ 38' 7'' 50'''$, TH will be to TK in the subduplicate ratio

“ of the number 9,0827646 to the number 10,827646, that
 “ is, as 18,6524761 to 19,6524761. And, therefore, TH is
 “ to HK as 18,6524761 to 1; that is, as the motion of the
 “ sun in a fidereal year to the mean motion of the node 19°
 “ $18' 1'' 23\frac{2}{3}''$.

“ But if the mean motion of the moon's nodes in 20 Julian
 “ years is $386^{\circ} 50' 15''$, as is collected from the observations
 “ made use of in the theory of the moon, the mean motion
 “ of the nodes in one fidereal year will be $19^{\circ} 20' 31'' 58'''$.
 “ And TH will be to HK as 360° to $19^{\circ} 20' 31'' 58'''$; that is,
 “ as 18,61214 to 1: and from hence the mean horary motion
 “ of the nodes in the quadratures will come out $16'' 18''' 48^{iv}$.
 “ And the greatest equation of the nodes in the octants will
 “ be $1^{\circ} 29' 57''$.”

PROPOSITION XXXIV. PROBLEM XV.

To find the horary variation of the inclination of the moon's orbit to the plane of the ecliptic.

Let A and a (Pl. 13, Fig. 2) represent the syzygies; Q and q the quadratures; N and n the nodes; P the place of the moon in its orbit; p the orthographic projection of that place upon the plane of the ecliptic; and mTl the momentaneous motion of the nodes as above. If upon Tm we let fall the perpendicular PG, and joining pG we produce it till it meet Tl in g, and join also Pg, the angle PGp will be the inclination of the moon's orbit to the plane of the ecliptic when the moon is in P; and the angle Pgp will be the inclination of the same after a small moment of time is elapsed; and therefore the angle GPg will be the momentaneous variation of the inclination. But this angle GPg is to the angle GTg as TG to PG and Pp to PG conjunctly. And, therefore, if for the moment of time we assume an hour, since the angle GTg (by prop. 30) is to the angle $33'' 10''' 33^{iv}$. as $IT \times PG \times AZ$ to AT^3 , the angle GPg (or the horary variation of the inclination) will be to the angle $33'' 10''' 33^{iv}$. as $IT \times AZ \times TG \times \frac{Pp}{PG}$ to AT^3 . Q.E.I.

And thus it would be if the moon was uniformly revolved in a circular orbit. But if the orbit is elliptical, the mean mo-

tion of the nodes will be diminished in proportion of the lesser axis to the greater, as we have shewn above; and the variation of the inclination will be also diminished in the same proportion.

COR. 1. Upon Nn erect the perpendicular TF, and let pM be the horary motion of the moon in the plane of the ecliptic; upon QT let fall the perpendiculars pK, Mk, and produce them till they meet TF in H and h; then IT will be to AT as Kk to Mp; and TG to Hp as TZ to AT; and, therefore, IT $\times$ TG will be equal to $\frac{Kk \times Hp \times TZ}{Mp}$, that is, equal

to the area HpMh multiplied into the ratio $\frac{TZ}{Mp}$; and therefore the horary variation of the inclination will be to $33'' 10''' 33^{iv}$. as the area HpMh multiplied into $AZ \times \frac{TZ}{Mp} \times \frac{Pp}{PG}$ to AT^3 .

COR. 2. And, therefore, if the earth and nodes were after every hour drawn back from their new and instantly restored to their old places, so as their situation might continue given for a whole periodic month together, the whole variation of the inclination during that month would be to $33'' 10''' 33^{iv}$. as the aggregate of all the areas HpMh, generated in the time of one revolution of the point p (with due regard in summing to their proper signs + and -), multiplied into $AZ \times TZ \times \frac{Pp}{PG}$ to $Mp \times AT^3$; that is, as the whole circle

QAqa multiplied into $AZ \times TZ \times \frac{Pp}{PG}$ to $Mp \times AT^3$, that is, as the circumference QAqa multiplied into $AZ \times TZ \times \frac{Pp}{PG}$ to $2Mp \times AT^3$.

COR. 3. And, therefore, in a given position of the nodes, the mean horary variation, from which, if uniformly continued through the whole month, that mensrual variation might be generated, is to $33'' 10''' 33^{iv}$. as $AZ \times TZ \times \frac{Pp}{PG}$ to $2AT^3$, or as $Pp \times \frac{AZ \times TZ}{\frac{1}{2}AT}$ to $PG \times 4AT$; that is

(because Pp is to PG as the sine of the aforefaid inclination to the radius, and $\frac{AZ \times TZ}{\frac{1}{2}AT}$ to 4AT as the sine of double the angle ATn to four times the radius), as the sine of the same inclination multiplied into the sine of double the distance of the nodes from the sun to four times the square of the radius.

COR. 4. Seeing the horary variation of the inclination, when the nodes are in the quadratures, is (by this prop.) to the angle $33'' 10''' 33^{iv.}$ as $IT \times AZ \times TG \times \frac{Pp}{PG}$ to AT^3 , that is, as $\frac{IT \times TG}{\frac{1}{2}AT} \times \frac{Pp}{PG}$ to 2AT, that is, as the sine of double the distance of the moon from the quadratures multiplied into $\frac{Pp}{PG}$ to twice the radius, the sum of all the horary variations during the time that the moon, in this situation of the nodes, passes from the quadrature to the syzygy (that is, in the space of $177\frac{1}{6}$ hours) will be to the sum of as many angles $33'' 10''' 33^{iv.}$ or $5878''$, as the sum of all the sines of double the distance of the moon from the quadratures multiplied into $\frac{Pp}{PG}$ to the sum of as many diameters; that is, as the diameter multiplied into $\frac{Pp}{PG}$ to the circumference; that is, if the inclination be $5^\circ 1'$, as $7 \times \frac{87\frac{1}{2}}{10000}$ to 22, or as 278 to 10000. And, therefore, the whole variation, composed out of the sum of all the horary variations in the aforefaid time, is $163''$, or $2' 43''$.

PROPOSITION XXXV. PROBLEM XVI.

To a given time to find the inclination of the moon's orbit to the plane of the ecliptic.

Let AD (Pl. 14, Fig. 1) be the sine of the greatest inclination, and AB the sine of the least. Bisect BD in C; and round the centre C, with the interval BC, describe the circle BGD. In AC take CE in the same proportion to EB as EB to twice BA. And if to the time given, we set off the angle AEG equal to double the distance of the nodes from the qua-

dratures, and upon AD let fall the perpendicular GH, AH will be the fine of the inclination required.

For GE^2 is equal to $GH^2 + HE^2 = BHD + HE^2 = HBD + HE^2 - BH^2 = HBD + BE^2 - 2BH \times BE = BE^2 + 2EC \times BH = 2EC \times AB + 2EC \times BH = 2EC \times AH$; wherefore since $2EC$ is given, GE^2 will be as AH . Now let Aeg represent double the distance of the nodes from the quadratures, in a given moment of time after, and the arc Gg, on account of the given angle GEg, will be as the distance GE. But Hh is to Gg as GH to GC, and, therefore, Hh is as the rectangle $GH \times Gg$, or $GH \times GE$, that is, as $\frac{GH}{GE} \times GE^2$, or $\frac{GH}{GE} \times AH$; that is, as AH and the fine of the angle AEG conjunctly. If, therefore, in any one case, AH be the fine of inclination, it will increase by the same increments as the fine of inclination doth, by cor. 3 of the preceding prop., and therefore will always continue equal to that fine. But when the point G falls upon either point B or D, AH is equal to this fine, and therefore remains always equal thereto. Q.E.D.

In this demonstration I have supposed that the angle BEG, representing double the distance of the nodes from the quadratures, increaseth uniformly; for I cannot descend to every minute circumstance of inequality. Now suppose that BEG is a right angle, and that Gg is in this case the horary increment of double the distance of the nodes from the sun; then, by cor. 3 of the last prop. the horary variation of the inclination in the same case will be to $33'' 10''' 33^{iv}$. as the rectangle of AH , the fine of the inclination, into the fine of the right angle BEG, double the distance of the nodes from the sun, to four times the square of the radius; that is, as AH , the fine of the mean inclination, to four times the radius; that is, seeing the mean inclination is about $5^\circ 8\frac{1}{2}'$, as its fine 896 to 40000, the quadruple of the radius, or as 224 to 10000. But the whole variation corresponding to BD, the difference of the fines, is to this horary variation as the diameter BD to the arc Gg, that is, conjunctly as the diameter BD to the semi-circumference BGD, and as the time of $2079\frac{7}{10}$ hours, in which

the node proceeds from the quadratures to the syzygies, to one hour, that is, as 7 to 11, and $2079\frac{7}{10}$ to 1. Wherefore, compounding all these proportions, we shall have the whole variation BD to $33'' 10''' 33^{iv}$. as $224 \times 7 \times 2079\frac{7}{10}$ to 110000, that is, as 29645 to 1000; and from thence that variation BD will come out $16' 23\frac{7}{2}''$.

And this is the greatest variation of the inclination, abstracting from the situation of the moon in its orbit; for if the nodes are in the syzygies, the inclination suffers no change from the various positions of the moon. But if the nodes are in the quadratures, the inclination is less when the moon is in the syzygies than when it is in the quadratures by a difference of $2' 43''$, as we shewed in cor. 4 of the preceding prop.; and the whole mean variation BD, diminished by $1' 21\frac{1}{2}''$, the half of this excess, becomes $15' 2''$, when the moon is in the quadratures; and increased by the same, becomes $17' 45''$ when the moon is in the syzygies. If, therefore, the moon be in the syzygies, the whole variation in the passage of the nodes from the quadratures to the syzygies will be $17' 45''$; and, therefore, if the inclination be $5^\circ 17' 20''$, when the nodes are in the syzygies, it will be $4^\circ 59' 35''$ when the nodes are in the quadratures and the moon in the syzygies. The truth of all which is confirmed by observations.

Now if the inclination of the orbit should be required when the moon is in the syzygies, and the nodes any where between them and the quadratures, let AB be to AD as the sine of $4^\circ 59' 35''$ to the sine of $5^\circ 17' 20''$, and take the angle AEG equal to double the distance of the nodes from the quadratures; and AH will be the sine of the inclination desired. To this inclination of the orbit the inclination of the same is equal, when the moon is 90° distant from the nodes. In other situations of the moon, this menstrual inequality, to which the variation of the inclination is obnoxious in the calculus of the moon's latitude, is balanced, and in a manner took off, by the menstrual inequality of the motion of the nodes (as we said before), and therefore may be neglected in the computation of the said latitude.

SCHOLIUM.

By these computations of the lunar motions I was willing to shew that by the theory of gravity the motions of the moon could be calculated from their physical causes. By the same theory I moreover found that the annual equation of the mean motion of the moon arises from the various dilatations which the orbit of the moon suffers from the action of the sun, according to cor. 6, prop. 66, book 1. The force of this action is greater in the perigeon sun, and dilates the moon's orbit; in the apogee sun it is less, and permits the orbit to be again contracted. The moon moves slower in the dilated and faster in the contracted orbit; and the annual equation, by which this inequality is regulated, vanishes in the apogee and perigee of the sun. In the mean distance of the sun from the earth it arises to about $11' 50''$; in other distances of the sun it is proportional to the equation of the sun's centre, and is added to the mean motion of the moon, while the earth is passing from its aphelion to its perihelion, and subtracted while the earth is in the opposite semicircle. Taking for the radius of the *orbis magnus* 1000, and $16\frac{7}{8}$ for the earth's eccentricity, this equation, when of the greatest magnitude, by the theory of gravity comes out $11' 49''$. But the eccentricity of the earth seems to be something greater, and with the eccentricity this equation will be augmented in the same proportion. Suppose the eccentricity $16\frac{11}{12}$, and the greatest equation will be $11' 51''$.

Farther; I found that the apogee and nodes of the moon move faster in the perihelion of the earth, where the force of the sun's action is greater, than in the aphelion thereof, and that in the reciprocal triplicate proportion of the earth's distance from the sun; and hence arise annual equations of those motions proportional to the equation of the sun's centre. Now the motion of the sun is in the reciprocal duplicate proportion of the earth's distance from the sun; and the greatest equation of the centre which this inequality generates is $1^{\circ} 56' 20''$, corresponding to the above-mentioned eccentricity of the sun, $16\frac{11}{12}$. But if the motion of the sun had been in the reciprocal triplicate proportion of the distance, this ine-

quality would have generated the greatest equation $2^{\circ} 54' 30''$; and therefore the greatest equations which the inequalities of the motions of the moon's apogee and nodes do generate are to $2^{\circ} 54' 30''$ as the mean diurnal motion of the moon's apogee and the mean diurnal motion of its nodes are to the mean diurnal motion of the sun. Whence the greatest equation of the mean motion of the apogee comes out $19' 43''$, and the greatest equation of the mean motion of the nodes $9' 24''$. The former equation is added, and the latter subtracted, while the earth is passing from its perihelion to its aphelion, and contrariwise when the earth is in the opposite semicircle.

By the theory of gravity I likewise found that the action of the sun upon the moon is something greater when the transverse diameter of the moon's orbit passeth through the sun than when the same is perpendicular upon the line which joins the earth and the sun; and therefore the moon's orbit is something larger in the former than in the latter case. And hence arises another equation of the moon's mean motion, depending upon the situation of the moon's apogee in respect of the sun, which is in its greatest quantity when the moon's apogee is in the octants of the sun, and vanishes when the apogee arrives at the quadratures or syzygies; and it is added to the mean motion while the moon's apogee is passing from the quadrature of the sun to the syzygy, and subtracted while the apogee is passing from the syzygy to the quadrature. This equation, which I shall call the semi-annual, when greatest in the octants of the apogee, arises to about $3' 45''$, so far as I could collect from the phenomena: and this is its quantity in the mean distance of the sun from the earth. But it is increased and diminished in the reciprocal triplicate proportion of the sun's distance, and therefore is nearly $3' 34''$ when that distance is greatest, and $3' 56''$ when least. But when the moon's apogee is without the octants, it becomes less, and is to its greatest quantity as the sine of double the distance of the moon's apogee from the nearest syzygy or quadrature to the radius.

By the same theory of gravity, the action of the sun upon the moon is something greater when the line of the moon's

nodes passes through the sun than when it is at right angles with the line which joins the sun and the earth; and hence arises another equation of the moon's mean motion, which I shall call the second semi-annual; and this is greatest when the nodes are in the octants of the sun, and vanishes when they are in the syzygies or quadratures; and in other positions of the nodes is proportional to the sine of double the distance of either node from the nearest syzygy or quadrature. And it is added to the mean motion of the moon, if the sun is *in antecedentia*, to the node which is nearest to him, and subtracted if *in consequentia*; and in the octants, where it is of the greatest magnitude, it arises to $47''$ in the mean distance of the sun from the earth, as I find from the theory of gravity. In other distances of the sun, this equation, greatest in the octants of the nodes, is reciprocally as the cube of the sun's distance from the earth; and therefore in the sun's perigee it comes to about $49''$, and in its apogee to about $45''$.

By the same theory of gravity, the moon's apogee goes forward at the greatest rate when it is either in conjunction with or in opposition to the sun, but in its quadratures with the sun it goes backward; and the eccentricity comes, in the former case, to its greatest quantity; in the latter to its least, by cor. 7, 8, and 9, prop. 66, book 1. And those inequalities, by the corollaries we have named, are very great, and generate the principal, which I call the semi-annual equation of the apogee; and this semi-annual equation in its greatest quantity comes to about $12^{\circ} 18''$, as nearly as I could collect from the phenomena. Our countryman *Horrox* was the first who advanced the theory of the moon's moving in an ellipsis about the earth placed in its lower focus. Dr. *Halley* improved the notion, by putting the centre of the ellipsis in an epicycle whose centre is uniformly revolved about the earth; and from the motion in this epicycle the mentioned inequalities in the progress and regress of the apogee, and in the quantity of eccentricity, do arise. Suppose the mean distance of the moon from the earth to be divided into 100000 parts, and let T (Pl. 14, Fig. 2) represent the earth, and TC the moon's mean eccentricity of 5505 such parts. Produce TC to B,

so as CB may be the sine of the greatest semi-annual equation $12^{\circ} 18'$ to the radius TC; and the circle BDA described about the centre C, with the interval CB, will be the epicycle spoken of, in which the centre of the moon's orbit is placed, and revolved according to the order of the letters BDA. Set off the angle BCD equal to twice the annual argument, or twice the distance of the sun's true place from the place of the moon's apogee once equated, and CTD will be the semi-annual equation of the moon's apogee, and TD the eccentricity of its orbit, tending to the place of the apogee now twice equated. But, having the moon's mean motion, the place of its apogee, and its eccentricity, as well as the longer axis of its orbit 200000, from these *data* the true place of the moon in its orbit, together with its distance from the earth, may be determined by the methods commonly known.

In the perihelion of the earth, where the force of the sun is greatest, the centre of the moon's orbit moves faster about the centre C than in the aphelion, and that in the reciprocal triplicate proportion of the sun's distance from the earth. But, because the equation of the sun's centre is included in the annual argument, the centre of the moon's orbit moves faster in its epicycle BDA, in the reciprocal duplicate proportion of the sun's distance from the earth. Therefore that it may move yet faster in the reciprocal simple proportion of the distance, suppose that from D, the centre of the orbit, a right line DE is drawn, tending towards the moon's apogee once equated, that is, parallel to TC; and set off the angle EDF equal to the excess of the aforesaid annual argument above the distance of the moon's apogee from the sun's perigee in *consequentia*; or, which comes to the same thing, take the angle CDF equal to the complement of the sun's true anomaly to 360° ; and let DF be to DC as twice the eccentricity of the *orbis magnus* to the sun's mean distance from the earth, and the sun's mean diurnal motion from the moon's apogee to the sun's mean diurnal motion from its own apogee conjunctly, that is, as $33\frac{1}{8}$ to 1000, and $52^{\circ} 27' 16''$ to $59^{\circ} 8' 10''$ conjunctly, or as 3 to 100; and imagine the centre of

the moon's orbit placed in the point F to be revolved in an epicycle whose centre is D, and radius DF, while the point D moves in the circumference of the circle DABD; for by this means the centre of the moon's orbit comes to describe a certain curve line about the centre C, with a velocity which will be almost reciprocally as the cube of the sun's distance from the earth, as it ought to be.

The calculus of this motion is difficult, but may be rendered more easy by the following approximation. Assuming, as above, the moon's mean distance from the earth of 100000 parts, and the eccentricity TC of 5505 such parts, the line CB or CD will be found $1172\frac{3}{4}$, and DF $35\frac{1}{2}$ of those parts; and this line DF at the distance TC subtends the angle at the earth, which the removal of the centre of the orbit from the place D to the place F generates in the motion of this centre; and double this line DF in a parallel position, at the distance of the upper focus of the moon's orbit from the earth, subtends at the earth the same angle as DF did before, which that removal generates in the motion of this upper focus; but at the distance of the moon from the earth this double line 2DF at the upper focus, in a parallel position to the first line DF, subtends an angle at the moon, which the said removal generates in the motion of the moon, which angle may be therefore called the second equation of the moon's centre; and this equation, in the mean distance of the moon from the earth, is nearly as the sine of the angle which that line DF contains with the line drawn from the point F to the moon, and when in its greatest quantity amounts to $2' 25''$. But the angle which the line DF contains with the line drawn from the point F to the moon is found either by subtracting the angle EDF from the mean anomaly of the moon, or by adding the distance of the moon from the sun to the distance of the moon's apogee from the apogee of the sun; and as the radius to the sine of the angle thus found, so is $2' 25''$ to the second equation of the centre: to be added, if the forementioned sum be less than a semicircle; to be subtracted, if greater. And from the moon's place in its orbit thus corrected its longitude may be found in the syzygies of the luminaries.

The atmosphere of the earth to the height of 35 or 40 miles refracts the sun's light. This refraction scatters and spreads the light over the earth's shadow; and the dissipated light near the limits of the shadow dilates the shadow. Upon which account, to the diameter of the shadow, as it comes out by the parallax, I add 1 or $1\frac{1}{3}$ minute in lunar eclipses.

But the theory of the moon ought to be examined and proved from the phænomena, first in the syzygies, then in the quadratures, and last of all in the octants; and whoever pleases to undertake the work will find it not amiss to assume the following mean motions of the sun and moon at the Royal Observatory of *Greenwich*, to the last day of *December* at noon, anno 1700, O.S. viz. The mean motion of the sun $\approx 20^{\circ} 43' 40''$, and of its apogee $\approx 7^{\circ} 44' 30''$; the mean motion of the moon $\approx 15^{\circ} 21' 00''$; of its apogee, $\times 8^{\circ} 20' 00''$; and of its ascending node $\Omega 27^{\circ} 24' 20''$; and the difference of meridians betwixt the Observatory at *Greenwich* and the Royal Observatory at *Paris*, $0^h. 9' 20''$: but the mean motion of the moon and of its apogee is not yet obtained with sufficient accuracy.

PROPOSITION XXXVI. PROBLEM XVII.

To find the force of the sun to move the sea.

The sun's force ML or PT to disturb the motions of the moon, was (by prop. 25) in the moon's quadratures, to the force of gravity with us, as 1 to 638092,6; and the force TM — LM or 2PK in the moon's syzygies is double that quantity. But, descending to the surface of the earth, these forces are diminished in proportion of the distances from the centre of the earth, that is, in the proportion of $60\frac{1}{2}$ to 1; and therefore the former force on the earth's surface is to the force of gravity as 1 to 38604600; and by this force the sea is depressed in such places as are 90 degrees distant from the sun. But by the other force, which is twice as great, the sea is raised not only in the places directly under the sun, but in those also which are directly opposed to it; and the sum of these forces is to the force of gravity as 1 to 12868200. And because the same force excites the same motion, whether it depresses the waters in those places which are 90 degrees distant from the sun, or raises them in the places which

are directly under and directly opposed to the sun, the aforesaid sun will be the total force of the sun to disturb the sea, and will have the same effect as if the whole was employed in raising the sea in the places directly under and directly opposed to the sun, and did not act at all in the places which are 90 degrees removed from the sun.

And this is the force of the sun to disturb the sea in any given place, where the sun is at the same time both vertical, and in its mean distance from the earth. In other positions of the sun, its force to raise the sea is as the versed sine of double its altitude above the horizon of the place directly, and the cube of the distance from the earth reciprocally.

COR. Since the centrifugal force of the parts of the earth, arising from the earth's diurnal motion, which is to the force of gravity as 1 to 289, raises the waters under the equator to a height exceeding that under the poles by 85472 *Paris* feet, as above, in prop. 19, the force of the sun, which we have now shewn to be to the force of gravity as 1 to 12868200, and therefore is to that centrifugal force as 289 to 12868200, or as 1 to 44527, will be able to raise the waters in the places directly under and directly opposed to the sun to a height exceeding that in the places which are 90 degrees removed from the sun only by 1 *Paris* foot and $113\frac{1}{30}$ inches; for this measure is to the measure of 85472 feet as 1 to 44527.

PROPOSITION XXXVII. PROBLEM XVIII.

To find the force of the moon to move the sea.

The force of the moon to move the sea is to be deduced from its proportion to the force of the sun, and this proportion is to be collected from the proportion of the motions of the sea, which are the effects of those forces. Before the mouth of the river *Avon*, three miles below *Bristol*, the height of the ascent of the water in the vernal and autumnal syzygies of the luminaries (by the observations of *Samuel Sturmy*) amounts to about 45 feet, but in the quadratures to 25 only. The former of those heights arises from the sum of the aforesaid forces, the latter from their difference. If, therefore, S and L are supposed to represent respectively the forces of the sun and moon while they are in the equator, as well as in

their mean distances from the earth, we shall have $L + S$ to $L - S$ as 45 to 25, or as 9 to 5.

At *Plymouth* (by the observations of *Samuel Coleprefs*) the tide in its mean height rises to about 16 feet, and in the spring and autumn the height thereof in the syzygies may exceed that in the quadratures by more than 7 or 8 feet. Suppose the greatest difference of those heights to be 9 feet, and $L + S$ will be to $L - S$ as $20\frac{1}{2}$ to $11\frac{1}{2}$, or as 41 to 23; a proportion that agrees well enough with the former. But because of the great tide at *Bristol*, we are rather to depend upon the observations of *Sturmy*; and, therefore, till we procure something that is more certain, we shall use the proportion of 9 to 5.

But because of the reciprocal motions of the waters, the greatest tides do not happen at the times of the syzygies of the luminaries, but, as we have said before, are the third in order after the syzygies; or (reckoning from the syzygies) follow next after the third appulse of the moon to the meridian of the place after the syzygies; or rather (as *Sturmy* observes) are the third after the day of the new or full moon, or rather nearly after the twelfth hour from the new or full moon, and therefore fall nearly upon the forty-third hour after the new or full of the moon. But in this port they fall out about the seventh hour after the appulse of the moon to the meridian of the place; and therefore follow next after the appulse of the moon to the meridian, when the moon is distant from the sun, or from opposition with the sun by about 18 or 19 degrees in *consequentia*. So the summer and winter seasons come not to their height in the solstices themselves, but when the sun is advanced beyond the solstices by about a tenth part of its whole course, that is, by about 36 or 37 degrees. In like manner, the greatest tide is raised after the appulse of the moon to the meridian of the place, when the moon has passed by the sun, or the opposition thereof, by about a tenth part of the whole motion from one greatest tide to the next following greatest tide. Suppose that distance about $18\frac{1}{2}$ degrees; and the sun's force in this distance of the moon from the syzygies and quadratures will be of less mo-

ment to augment and diminish that part of the motion of the sea which proceeds from the motion of the moon than in the syzygies and quadratures themselves in the proportion of the radius to the cosine of double this distance, or of an angle of 37 degrees; that is, in proportion of 10000000 to 7986355; and, therefore, in the preceding analogy, in place of S we must put 0,7986355S.

But farther; the force of the moon in the quadratures must be diminished, on account of its declination from the equator; for the moon in those quadratures, or rather in $18\frac{1}{2}$ degrees past the quadratures, declines from the equator by about $22^{\circ} 13'$; and the force of either luminary to move the sea is diminished as it declines from the equator nearly in the duplicate proportion of the cosine of the declination; and therefore the force of the moon in those quadratures is only 0,8570327L; whence we have $L + 0,7986355S$ to $0,8570327L - 0,7986355S$ as 9 to 5.

Farther yet; the diameters of the orbit in which the moon should move, setting aside the consideration of eccentricity, are one to the other as 69 to 70; and therefore the moon's distance from the earth in the syzygies is to its distance in the quadratures, *ceteris paribus*, as 69 to 70; and its distances, when $18\frac{1}{2}$ degrees advanced beyond the syzygies, where the greatest tide was excited, and when $18\frac{1}{2}$ degrees passed by the quadratures, where the least tide was produced, are to its mean distance as 69,098747 and 69,897345 to $69\frac{1}{2}$. But the force of the moon to move the sea is in the reciprocal triplicate proportion of its distance; and therefore its forces, in the greatest and least of those distances, are to its force in its mean distance as 0,9830427 and 1,017522 to 1. From whence we have $1,017522L \times 0,7986355S$ to $0,9830427 \times 0,8570327L - 0,7986355S$ as 9 to 5; and S to L as 1 to 4,4815. Wherefore, since the force of the sun is to the force of gravity as 1 to 12868200, the moon's force will be to the force of gravity as 1 to 2871400.

COR. 1. Since the waters excited by the sun's force rise to the height of a foot and $11\frac{1}{30}$ inches, the moon's force will raise the same to the height of 8 feet and $7\frac{5}{8}$ inches; and the

joint forces of both will raise the same to the height of $10\frac{1}{2}$ feet; and when the moon is in its perigee to the height of $12\frac{1}{2}$ feet, and more, especially when the wind sets the same way as the tide. And a force of that quantity is abundantly sufficient to excite all the motions of the sea, and agrees well with the proportion of those motions; for in such seas as lie free and open from east to west, as in the *Pacific* sea, and in those tracts of the *Atlantic* and *Ethiopic* seas which lie without the tropics, the waters commonly rise to 6, 9, 12, or 15 feet; but in the *Pacific* sea, which is of a greater depth, as well as of a larger extent, the tides are said to be greater than in the *Atlantic* and *Ethiopic* sea; for to have a full tide raised, an extent of sea from east to west is required of no less than 90 degrees. In the *Ethiopic* sea, the waters rise to a less height within the tropics than in the temperate zones, because of the narrowness of the sea between *Africa* and the southern parts of *America*. In the middle of the open sea the waters cannot rise without falling together, and at the same time, upon both the eastern and western shores, when notwithstanding, in our narrow seas, they ought to fall on those shores by alternate turns; upon which account there is commonly but a small flood and ebb in such islands as lie far distant from the continent. On the contrary, in some ports, where to fill and empty the bays alternately the waters are with great violence forced in and out through shallow channels, the flood and ebb must be greater than ordinary; as at *Plymouth* and *Chepstow Bridge* in *England*, at the mountains of *St. Michael*, at the town of *Auranches* in *Normandy*, and at *Cambaia* and *Pegu* in the *East Indies*. In these places the sea is hurried in and out with such violence, as sometimes to lay the shores under water, sometimes to leave them dry, for many miles. Nor is this force of the influx and efflux to be broke till it has raised and depressed the waters to 30, 40, or 50 feet and above. And a like account is to be given of long and shallow channels or straits, such as the *Magellanic* straits, and those channels which environ *England*. The tide in such ports and straits, by the violence of the influx and efflux, is augmented above measure. But on such shores as lie towards

the deep and open sea with a steep descent, where the waters may freely rise and fall without that precipitation of influx and efflux, the proportion of the tides agrees with the forces of the sun and moon.

COR. 2. Since the moon's force to move the sea is to the force of gravity as 1 to 2871400, it is evident that this force is far less than to appear sensibly in statical or hydrostatical experiments, or even in those of pendulums. It is in the tides only that this force shews itself by any sensible effect.

COR. 3. Because the force of the moon to move the sea is to the like force of the sun as 4,4815 to 1, and those forces (by cor. 14, prop. 66, book 1) are as the densities of the bodies of the sun and moon and the cubes of their apparent diameters conjunctly, the density of the moon will be to the density of the sun as 4,4815 to 1 directly, and the cube of the moon's diameter to the cube of the sun's diameter inversely; that is (seeing the mean apparent diameters of the moon and sun are $31' 16\frac{1}{2}''$, and $32' 12''$), as 4891 to 1000. But the density of the sun was to the density of the earth as 1000 to 4000; and therefore the density of the moon is to the density of the earth as 4891 to 4000, or as 11 to 9. Therefore the body of the moon is more dense and more earthly than the earth itself.

COR. 4. And since the true diameter of the moon (from the observations of astronomers) is to the true diameter of the earth as 100 to 365, the mass of matter in the moon will be to the mass of matter in the earth as 1 to 39,788.

COR. 5. And the accelerative gravity on the surface of the moon will be about three times less than the accelerative gravity on the surface of the earth.

COR. 6. And the distance of the moon's centre from the centre of the earth will be to the distance of the moon's centre from the common centre of gravity of the earth and moon as 40,788 to 39,788.

COR. 7. And the mean distance of the centre of the moon from the centre of the earth will be (in the moon's octants) nearly $60\frac{2}{3}$ of the greatest semidiameters of the earth; for the greatest semidiameters of the earth was 19658600 *Paris*

feet, and the mean distance of the centres of the earth and moon, consisting of $60\frac{1}{2}$ such semidiameters, is equal to 1187379440 feet. And this distance (by the preceding cor.) is to the distance of the moon's centre from the common centre of gravity of the earth and moon as 40,788 to 39,788; which latter distance, therefore, is 1158268534 feet. And since the moon, in respect of the fixed stars, performs its revolution in $27^d. 7^h. 43\frac{1}{3}'$, the versed sine of that angle which the moon in a minute of time describes is 12752341 to the radius 1000,000000,000000; and as the radius is to this versed sine, so are 1158268534 feet to 14,7706353 feet. The moon, therefore, falling towards the earth by that force which retains it in its orbit, would in one minute of time describe 14,7706353 feet; and if we augment this force in the proportion of $178\frac{29}{40}$ to $177\frac{29}{40}$ we shall have the total force of gravity at the orbit of the moon, by cor. prop. 3; and the moon falling by this force, in one minute of time would describe 14,8538067 feet. And at the 60th part of the distance of the moon from the earth's centre, that is, at the distance of 197896573 feet from the centre of the earth, a body falling by its weight, would, in one second of time, likewise describe 14,8538067 feet. And, therefore, at the distance of 19615800, which compose one mean semidiameter of the earth, a heavy body would describe in falling 15,11175, or 15 feet 1 inch and $4\frac{1}{11}$ lines, in the same time. This will be the descent of bodies in the latitude of 45 degrees. And by the foregoing table, to be found under prop. 20, the descent in the latitude of *Paris* will be a little greater by an excess of about $\frac{3}{4}$ parts of a line. Therefore, by this computation, heavy bodies in the latitude of *Paris* falling *in vacuo* will describe 15 *Paris* feet 1 inch $4\frac{3}{4}$ lines, very nearly, in one second of time. And if the gravity be diminished by taking away a quantity equal to the centrifugal force arising in that latitude from the earth's diurnal motion, heavy bodies falling there will describe in one second of time 15 feet 1 inch and $1\frac{1}{2}$ line. And with this velocity heavy bodies do really fall in the latitude of *Paris*, as we have shewn above in prop. 4 and 19.

COR. 8. The mean distance of the centres of the earth and moon in the syzygies of the moon is equal to 60 of the greatest semidiameters of the earth, subducting only about one 30th part of a semidiameter; and in the moon's quadratures the mean distance of the same centres is $60\frac{5}{8}$ such semidiameters of the earth; for these two distances are to the mean distance of the moon in the octants as 69 and 70 to $69\frac{1}{2}$, by prop. 28.

COR. 9. The mean distance of the centres of the earth and moon in the syzygies of the moon is 60 semidiameters of the earth, and a 10th part of one semidiameter; and in the moon's quadratures the mean distance of the same centres is 61 mean semidiameters of the earth, subducting one 30th part of one semidiameter.

COR. 10. In the moon's syzygies its mean horizontal parallax in the latitudes of 0, 30, 38, 45, 52, 60, 90 degrees is $57' 20''$, $57' 16''$, $57' 14''$, $57' 12''$, $57' 10''$, $57' 8''$, $57' 4''$, respectively.

In these computations I do not consider the magnetic attraction of the earth, whose quantity is very small and unknown: if the quantity should ever be found out, and the measures of degrees upon the meridian, the lengths of isochronous pendulums in different parallels, the laws of the motions of the sea, and the moon's parallax, with the apparent diameters of the sun and moon, should be more exactly determined from phenomena; we should then be enabled to bring this calculation to a greater accuracy.

PROPOSITION XXXVII. PROBLEM XIX,

To find the figure of the moon's body.

If the moon's body were fluid like our sea, the force of the earth to raise that fluid in the nearest and remotest parts would be to the force of the moon by which our sea is raised in the places under and opposite to the moon as the accelerative gravity of the moon towards the earth to the accelerative gravity of the earth towards the moon, and the diameter of the moon to the diameter of the earth conjunctly; that is, as 39,788 to 1, and 100 to 365 conjunctly, or as 1081 to 100. Wherefore since our sea, by the force of the moon, is raised to $8\frac{1}{2}$ feet, the lunar fluid would be raised by the

force of the earth to 93 feet; and upon this account the figure of the moon would be a spheroid, whose greatest diameter produced would pass through the centre of the earth, and exceed the diameters perpendicular thereto by 186 feet. Such a figure, therefore, the moon affects, and must have put on from the beginning. Q.E.I.

COR. Hence it is that the same face of the moon always respects the earth; nor can the body of the moon possibly rest in any other position, but would return always by a libratory motion to this situation; but those librations, however, must be exceedingly slow, because of the weakness of the forces which excite them; so that the face of the moon, which should be always obverted to the earth, may, for the reason assigned in prop. 17, be turned towards the other focus of the moon's orbit, without being immediately drawn back, and converted again towards the earth.

LEMMA I.

If $APEp$ (Pl. 14, Fig. 3) represent the earth uniformly dense, marked with the centre C , the poles P, p , and the equator AE ; and if about the centre C , with the radius CP , we suppose the sphere $Pape$ to be described, and QR to denote the plane on which a right line, drawn from the centre of the sun to the centre of the earth, insists at right angles; and farther suppose that the several particles of the whole exterior earth $PapAPEpE$, without the height of the said sphere, endeavour to recede towards this side and that side from the plane QR , every particle by a force proportional to its distance from that plane; I say, in the first place, that the whole force and efficacy of all the particles that are situate in AE , the circle of the equator, and disposed uniformly without the globe, encompassing the same after the manner of a ring, to wheel the earth about its centre, is to the whole force and efficacy of as many particles in that point A of the equator which is at the greatest distance from the plane QR , to wheel the earth about its centre with a little circular motion, as 1 to 2. And that circular motion will be performed about an axis lying in the common section of the equator and the plane QR .

For let there be described from the centre K , with the diameter IL , the semicircle $INLK$. Suppose the semi-circumference INL to be divided into innumerable equal parts, and from the several parts N to the diameter IL let fall the fines NM . Then the sums of the squares of all the fines NM will be equal to the sums of the squares of the fines KM , and both sums together will be equal to the sums of the squares of as many femidiameters KN ; and therefore the sum of the squares of all the fines NM will be but half so great as the sum of the squares of as many femidiameters KN .

Suppose, now, the circumference of the circle AE to be divided into the like number of little equal parts, and from every such part F a perpendicular FG to be let fall upon the plane QR , as well as the perpendicular AH from the point A . Then the force by which the particle F recedes from the plane QR will (by supposition) be as that perpendicular FG ; and this force, multiplied by the distance CG , will represent the power of the particle F to turn the earth round its centre. And, therefore, the power of a particle in the place F will be to the power of a particle in the place A as $FG \times GC$ to $AH \times HC$; that is, as FC^2 to AC^2 : and therefore the whole power of all the particles F , in their proper places F , will be to the power of the like number of particles in the place A as the sum of all the FC^2 to the sum of all the AC^2 ; that is (by what we have demonstrated before), as 1 to 2.

Q.E.D.

And because the action of those particles is exerted in the direction of lines perpendicularly receding from the plane QR , and that equally from each side of this plane, they will wheel about the circumference of the circle of the equator, together with the adherent body of the earth, round an axis which lies as well in the plane QR as in that of the equator.

LEMMA II.

The same things still supposed, I say, in the second place, that the total force or power of all the particles situated every where about the sphere to turn the earth about the said axis,

is to the whole force of the like number of particles, uniformly disposed round the whole circumference of the equator AE in the fashion of a ring, to turn the whole earth about with the like circular motion, as 2 to 5. (Pl. 14, Fig. 4.)

For let IK be any lesser circle parallel to the equator AE, and let L, l, be any two equal particles in this circle, situated without the sphere Pape; and if upon the plane QR, which is at right angles with a radius drawn to the sun, we let fall the perpendiculars LM, lm, the total forces by which these particles recede from the plane QR will be proportional to the perpendiculars LM, lm. Let the right line Ll be drawn parallel to the plane Pape, and bisect the same in X; and through the point X draw Nn parallel to the plane QR, and meeting the perpendiculars LM, lm, in N and n; and upon the plane QR let fall the perpendicular XY. And the contrary forces of the particles L and l to wheel about the earth contrariwise are as $LM \times MC$, and $lm \times mC$; that is, as $LN \times MC + NM \times MC$, and $ln \times mC - nm \times mC$; or $LN \times MC + NM \times MC$, and $LN \times mC - NM \times mC$, and $LN \times Mm - NM \times \overline{MC + mC}$, the difference of the two, is the force of both taken together to turn the earth round. The affirmative part of this difference $LN \times Mm$, or $2LN \times NX$, is to $2AH \times HC$, the force of two particles of the same size situated in A, as LX^2 to AC^2 ; and the negative part $NM \times \overline{MC + mC}$, or $2XY \times CY$, is to $2AH \times HC$, the force of the same two particles situated in A, as CX^2 to AC^2 . And therefore the difference of the parts, that is, the force of the two particles L and l, taken together, to wheel the earth about, is to the force of two particles, equal to the former and situated in the place A, to turn in like manner the earth round, as $LX^2 - CX^2$ to AC^2 . But if the circumference IK of the circle IK is supposed to be divided into an infinite number of little equal parts L, all the LX^2 will be to the like number of IX^2 as 1 to 2 (by lem. 1); and to the same number of AC^2 as IX^2 to $2AC^2$; and the same number of CX^2 to as many AC^2 as $2CX^2$ to $2AC^2$. Wherefore the united forces of all the particles in the circumference of the

circle IK are to the joint forces of as many particles in the place A as $IX^2 - 2CX^2$ to $2AC^2$; and therefore (by lem. 1) to the united forces of as many particles in the circumference of the circle AE as $IX^2 - 2CX^2$ to AC^2 .

Now, if Pp, the diameter of the sphere, is conceived to be divided into an infinite number of equal parts, upon which a like number of circles IK are supposed to insift, the matter in the circumference of every circle IK will be as IX^2 ; and therefore the force of that matter to turn the earth about will be as IX^2 into $IX^2 - 2CX^2$; and the force of the same matter, if it was situated in the circumference of the circle AE, would be as IX^2 into AC^2 . And therefore the force of all the particles of the whole matter situated without the sphere in the circumferences of all the circles is to the force of the like number of particles situated in the circumference of the greatest circle AE as all the IX^2 into $IX^2 - 2CX^2$ to as many IX^2 into AC^2 ; that is, as all the $AC^2 - CX^2$ into $AC^2 - 3CX^2$ to as many $AC^2 - CX^2$ into AC^2 ; that is, as all the $AC^4 - 4AC^2 \times CX^2 + 3CX^4$ to as many $AC^4 - AC^2 \times CX^2$; that is, as the whole fluent quantity, whose fluxion is $AC^4 - 4AC^2 \times CX^2 + 3CX^4$, to the whole fluent quantity, whose fluxion is $AC^4 - AC^2 \times CX^2$; and, therefore, by the method of fluxions, as $AC^4 \times CX - \frac{1}{2}AC^2 \times CX^3 + \frac{1}{3}CX^5$ to $AC^4 \times CX - \frac{1}{3}AC^2 \times CX^3$; that is, if for CX we write the whole Cp, or AC, as $\frac{1}{15}AC^5$ to $\frac{2}{3}AC^5$; that is, as 2 to 5. Q.E.D.

LEMMA III.

The same things still supposed, I say, in the third place, that the motion of the whole earth about the axis abovenamed arising from the motions of all the particles, will be to the motion of the afore said ring about the same axis in a proportion compounded of the proportion of the matter in the earth to the matter in the ring; and the proportion of three squares of the quadrantal arc of any circle to two squares of its diameter, that is, in the proportion of the matter to the matter, and of the number 925275 to the number 1000000.

For the motion of a cylinder revolved about its quiescent axis is to the motion of the inscribed sphere revolved together

with it as any four equal squares to three circles inscribed in three of those squares; and the motion of this cylinder is to the motion of an exceedingly thin ring furrounding both sphere and cylinder in their common contact as double the matter in the cylinder to triple the matter in the ring; and this motion of the ring, uniformly continued about the axis of the cylinder, is to the uniform motion of the same about its own diameter performed in the same periodic time, as the circumference of a circle to double its diameter.

HYPOTHESIS II.

If the other parts of the earth were taken away, and the remaining ring was carried alone about the sun in the orbit of the earth by the annual motion, while by the diurnal motion it was in the mean time revolved about its own axis inclined to the plane of the ecliptic by an angle of $23\frac{1}{2}$ degrees, the motion of the equinoctial points would be the same, whether the ring were fluid, or whether it consisted of a hard and rigid matter.

PROPOSITION XXXIX. PROBLEM XX.

To find the precession of the equinoxes.

The middle horary motion of the moon's nodes in a circular orbit, when the nodes are in the quadratures, was $16''\ 35'''$ $16^{\text{iv}}\ 36^{\text{v}}$; the half of which, $8''\ 17'''$ $38^{\text{iv}}\ 18^{\text{v}}$. (for the reasons above explained) is the mean horary motion of the nodes in such an orbit, which motion in a whole sidereal year becomes $20^{\circ}\ 11''\ 46''$. Because, therefore, the nodes of the moon in such an orbit would be yearly transferred $20^{\circ}\ 11'\ 46''$ *in antecedentia*; and if there were more moons, the motion of the nodes of every one (by cor. 16, prop. 66, book 1) would be as its periodic time; if upon the surface of the earth a moon was revolved in the time of a sidereal day, the annual motion of the nodes of this moon would be to $20^{\circ}\ 11'\ 46''$ as $23^{\text{h}}\ 56'$, the sidereal day, to $27^{\text{d}}\ 7^{\text{h}}\ 43'$, the periodic time of our moon, that is, as 1436 to 39343. And the same thing would happen to the nodes of a ring of moons encompassing the earth, whether these moons did not mutually touch each the other, or whether they were molten, and formed into a continued ring, or whether that ring should become rigid and inflexible.

Let us, then, suppose that this ring is in quantity of matter equal to the whole exterior earth $PapAPepE$, which lies without the sphere $Pape$ (see Fig. lem. 2); and because this sphere is to that exterior earth as aC^2 to $AC^2 - aC^2$, that is (seeing PC or aC the least semidiameter of the earth is to AC the greatest semidiameter of the same as 229 to 230), as 52441 to 459; if this ring encompassed the earth round the equator, and both together were revolved about the diameter of the ring, the motion of the ring (by lem. 3) would be to the motion of the inner sphere as 459 to 52441 and 1000000 to 925275 conjunctly, that is, as 4590 to 485223; and therefore the motion of the ring would be to the sum of the motions of both ring and sphere as 4590 to 489813. Wherefore if the ring adheres to the sphere, and communicates its motion to the sphere, by which its nodes or equinoctial points recede, the motion remaining in the ring will be to its former motion as 4590 to 489813; upon which account the motion of the equinoctial points will be diminished in the same proportion. Wherefore the annual motion of the equinoctial points of the body, composed of both ring and sphere, will be to the motion $20^\circ 11' 46''$ as 1436 to 39343 and 4590 to 489813 conjunctly, that is, as 100 to 292369. But the forces by which the nodes of a number of moons (as we explained above), and therefore by which the equinoctial points of the ring recede (that is, the forces $3IT$, in fig. prop. 30), are in the several particles as the distances of those particles from the plane QR ; and by these forces the particles recede from that plane; and therefore (by lem. 2) if the matter of the ring was spread all over the surface of the sphere, after the fashion of the figure $PapAPepE$, in order to make up that exterior part of the earth, the total force or power of all the particles to wheel about the earth round any diameter of the equator, and therefore to move the equinoctial points, would become less than before in the proportion of 2 to 5. Wherefore the annual regrefs of the equinoxes now would be to $20^\circ 11' 46''$ as 10 to 73092; that is, would be $9'' 56''' 50^{iv}$.

But because the plane of the equator is inclined to that of the ecliptic, this motion is to be diminished in the proportion

of the sine 91706 (which is the cosine of $23\frac{1}{2}$ deg.) to the radius 100000; and the remaining motion will now be $9'' 7''' 20^{iv}$. which is the annual precession of the equinoxes arising from the force of the sun.

But the force of the moon to move the sea was to the force of the sun nearly as 4,4815 to 1; and the force of the moon to move the equinoxes is to that of the sun in the same proportion. Whence the annual precession of the equinoxes proceeding from the force of the moon comes out $40'' 52''' 52^{iv}$. and the total annual precession arising from the united forces of both will be $50'' 00''' 12^{iv}$. the quantity of which motion agrees with the phenomena; for the precession of the equinoxes, by astronomical observations, is about $50''$ yearly.

If the height of the earth at the equator exceeds its height at the poles by more than $17\frac{1}{8}$ miles, the matter thereof will be more rare near the surface than at the centre; and the precession of the equinoxes will be augmented by the excess of height, and diminished by the greater rarity.

And now we have described the system of the sun, the earth, moon, and planets, it remains that we add something about the comets.

LEMMA IV.

That the comets are higher than the moon, and in the regions of the planets.

As the comets were placed by astronomers above the moon, because they were found to have no diurnal parallax, so their annual parallax is a convincing proof of their descending into the regions of the planets; for all the comets which move in a direct course according to the order of the signs, about the end of their appearance become more than ordinarily slow or retrograde, if the earth is between them and the sun; and more than ordinarily swift, if the earth is approaching to a heliocentric opposition with them; whereas, on the other hand, those which move against the order of the signs, towards the end of their appearance appear swifter than they ought to be, if the earth is between them and the sun; and slower, and perhaps retrograde, if the earth is in the other side of its orbit. And these appearances proceed chiefly from

the diverse situations which the earth acquires in the course of its motion, after the same manner as it happens to the planets, which appear sometimes retrograde, sometimes more slowly, and sometimes more swiftly, progressive, according as the motion of the earth falls in with that of the planet, or is directed the contrary way. If the earth move the same way with the comet, but, by an angular motion about the sun, so much swifter that right lines drawn from the earth to the comet converge towards the parts beyond the comet, the comet seen from the earth, because of its slower motion, will appear retrograde; and even if the earth is slower than the comet, the motion of the earth being subducted, the motion of the comet will at least appear retarded; but if the earth tends the contrary way to that of the comet, the motion of the comet will from thence appear accelerated; and from this apparent acceleration, or retardation, or regressive motion, the distance of the comet may be inferred in this manner. Let TQA , TQB , TQC (Pl. 15, Fig. 1), be three observed longitudes of the comet about the time of its first appearing, and TQF its last observed longitude before its disappearing. Draw the right line ABC , whose parts AB , BC , intercepted between the right lines QA and QB , QB and QC , may be one to the other as the two times between the three first observations. Produce AC to G , so as AG may be to AB as the time between the first and last observation to the time between the first and second; and join QG . Now if the comet did move uniformly in a right line, and the earth either stood still, or was likewise carried forwards in a right line by an uniform motion, the angle TQG would be the longitude of the comet at the time of the last observation. The angle, therefore, FQG , which is the difference of the longitude, proceeds from the inequality of the motions of the comet and the earth; and this angle, if the earth and comet move contrary ways, is added to the angle TQG , and accelerates the apparent motion of the comet; but if the comet move the same way with the earth, it is subtracted, and either retards the motion of the comet, or perhaps renders it retrograde, as we have but now explained. This angle, therefore, proceeding

chiefly from the motion of the earth, is justly to be esteemed the parallax of the comet; neglecting, to wit, some little increment or decrement that may arise from the unequal motion of the comet in its orbit; and from this parallax we thus deduce the distance of the comet. Let S (Pl. 15, Fig. 2) represent the sun, acT the *orbis magnus*, a the earth's place in the first observation, c the place of the earth in the third observation, T the place of the earth in the last observation, and Tt a right line drawn to the beginning of Aries. Set off the angle TTV equal to the angle TQF , that is, equal to the longitude of the comet at the time when the earth is in T ; join ac , and produce it to g , so as ag may be to ac as AG to AC ; and g will be the place at which the earth would have arrived in the time of the last observation, if it had continued to move uniformly in the right line ac . Wherefore, if we draw gr parallel to Tt , and make the angle rgV equal to the angle rqG , this angle rgV will be equal to the longitude of the comet seen from the place g , and the angle TVg will be the parallax which arises from the earth's being transferred from the place g into the place T ; and therefore V will be the place of the comet in the plane of the ecliptic. And this place V is commonly lower than the orb of Jupiter.

The same thing may be deduced from the incurvation of the way of the comets; for these bodies move almost in great circles, while their velocity is great; but about the end of their course, when that part of their apparent motion which arises from the parallax bears a greater proportion to their whole apparent motion, they continually deviate from those circles, and, when the earth goes to one side, they deviate to the other; and this deflexion, because of its corresponding with the motion of the earth, must arise chiefly from the parallax; and the quantity thereof is so considerable, as, by my computation, to place the disappearing comets a good deal lower than Jupiter. Whence it follows that, when they approach nearer to us in their perigees and perihelions, they often descend below the orbs of Mars and the inferior planets.

The near approach of the comets is farther confirmed from the light of their heads; for the light of a celestial body, illu-

minated by the sun, and receding to remote parts, is diminished in the quadruplicate proportion of the distance; to wit, in one duplicate proportion, on account of the increase of the distance from the sun, and in another duplicate proportion, on account of the decrease of the apparent diameter. Wherefore if both the quantity of light and the apparent diameter of a comet are given, its distance will be also given, by taking the distance of the comet to the distance of a planet in the direct proportion of their diameters and the reciprocal subduplicate proportion of their lights. Thus, in the comet of the year 1682, Mr. *Flamsted* observed with a telescope of 16 feet, and measured with a micrometer, the least diameter of its head, 2' 00; but the nucleus or star in the middle of the head scarcely amounted to the tenth part of this measure; and therefore its diameter was only 11" or 12"; but in the light and splendor of its head it surpassed that of the comet in the year 1680, and might be compared with the stars of the first or second magnitude. Let us suppose that Saturn with its ring was about four times more lucid; and because the light of the ring was almost equal to the light of the globe within, and the apparent diameter of the globe is about 21", and therefore the united light of both globe and ring would be equal to the light of a globe whose diameter is 30", it follows that the distance of the comet was to the distance of Saturn as 1 to $\sqrt{4}$ inversely, and 12" to 30 directly; that is, as 24 to 30, or 4 to 5. Again; the comet in the month of *April* 1665, as *Hevelius* informs us, excelled almost all the fixed stars in splendor, and even Saturn itself, as being of a much more vivid colour; for this comet was more lucid than that other which had appeared about the end of the preceding year, and had been compared to the stars of the first magnitude. The diameter of its head was about 6'; but the nucleus, compared with the planets by means of a telescope, was plainly less than Jupiter; and sometimes judged less, sometimes judged equal, to the globe of Saturn within the ring. Since, then, the diameters of the heads of the comets seldom exceed 8' or 12', and the diameter of the nucleus or central star is but about a tenth or perhaps fifteenth part of the diameter of

the head, it appears that these stars are generally of about the same apparent magnitude with the planets. But in regard that their light may be often compared with the light of Saturn, yea, and sometimes exceeds it, it is evident that all comets in their perihelions must either be placed below or not far above Saturn; and they are much mistaken who remove them almost as far as the fixed stars; for if it was so, the comets could receive no more light from our sun than our planets do from the fixed stars.

So far we have gone, without considering the obscuration which comets suffer from that plenty of thick smoke which encompasseth their heads, and through which the heads always shew dull, as through a cloud; for by how much the more a body is obscured by this smoke, by so much the more near it must be allowed to come to the sun, that it may vie with the planets in the quantity of light which it reflects. Whence it is probable that the comets descend far below the orb of Saturn, as we proved before from their parallax. But, above all, the thing is evinced from their tails, which must be owing either to the sun's light reflected by a smoke arising from them, and dispersing itself through the æther, or to the light of their own heads. In the former case, we must shorten the distance of the comets, lest we be obliged to allow that the smoke arising from their heads is propagated through such a vast extent of space, and with such a velocity and expansion, as will seem altogether incredible; in the latter case, the whole light of both head and tail is to be ascribed to the central nucleus. But, then, if we suppose all this light to be united and condensed within the disk of the nucleus, certainly the nucleus will by far exceed Jupiter itself in splendor, especially when it emits a very large and lucid tail. If, therefore, under a less apparent diameter, it reflects more light, it must be much more illuminated by the sun, and therefore much nearer to it; and the same argument will bring down the heads of comets sometimes within the orb of Venus, viz. when, being hid under the sun's rays, they emit such huge and splendid tails, like beams of fire, as sometimes they do; for if all that light was supposed to be gathered together into one

star, it would sometimes exceed not one Venus only, but a great many such united into one.

Lastly; the same thing is inferred from the light of the heads, which increases in the recess of the comets from the earth towards the sun, and decreases in their return from the sun towards the earth; for so the comet of the year 1665 (by the observations of *Hevelius*), from the time that it was first seen, was always losing of its apparent motion, and therefore had already passed its perigee; but yet the splendor of its head was daily increasing, till, being hid under the sun's rays, the comet ceased to appear. The comet of the year 1683 (by the observations of the same *Hevelius*), about the end of *July*, when it first appeared, moved at a very slow rate, advancing only about 40 or 45 minutes in its orb in a day's time; but from that time its diurnal motion was continually upon the increase, till *September 4*, when it arose to about 5 degrees; and therefore, in all this interval of time, the comet was approaching to the earth. Which is likewise proved from the diameter of its head, measured with a micrometer; for, *August 6*, *Hevelius* found it only 6' 05", including the coma, which, *September 2*, he observed to be 9' 07", and therefore its head appeared far less about the beginning than towards the end of the motion; though about the beginning, because nearer to the sun, it appeared far more lucid than towards the end, as the same *Hevelius* declares. Wherefore in all this interval of time, on account of its recess from the sun, it decreased in splendor, notwithstanding its access towards the earth. The comet of the year 1618, about the middle of *December*, and that of the year 1680, about the end of the same month, did both move with their greatest velocity, and were therefore then in their perigees; but the greatest splendor of their heads was seen two weeks before, when they had just got clear of the sun's rays; and the greatest splendor of their tails a little more early, when yet nearer to the sun. The head of the former comet (according to the observations of *Cysatus*), *December 1*, appeared greater than the stars of the first magnitude; and, *December 16* (then in the perigee), it was but little diminished in magnitude, but in the splendor and

brightness of its light a great deal. *January 7, Kepler*, being uncertain about the head, left off observing. *December 12*, the head of the latter comet was seen and observed by Mr. *Flamsted*, when but 9 degrees distant from the sun; which is scarcely to be done in a star of the third magnitude. *December 15 and 17*, it appeared as a star of the third magnitude, its lustre being diminished by the brightness of the clouds near the setting sun. *December 26*, when it moved with the greatest velocity, being almost in its perigee, it was less than the mouth of *Pegasus*, a star of the third magnitude. *January 3*, it appeared as a star of the fourth. *January 9*, as one of the fifth. *January 13*, it was hid by the splendor of the moon, then in her increase. *January 25*, it was scarcely equal to the stars of the seventh magnitude. If we compare equal intervals of time on one side and on the other from the perigee, we shall find that the head of the comet, which at both intervals of time was far, but yet equally, removed from the earth, and should have therefore shone with equal splendor, appeared brightest on the side of the perigee towards the sun, and disappeared on the other. Therefore, from the great difference of light in the one situation and in the other, we conclude the great vicinity of the sun and comet in the former; for the light of comets uses to be regular, and to appear greatest when the heads move fastest, and are therefore in their perigees; excepting in so far as it is increased by their nearness to the sun.

COR. 1. Therefore the comets shine by the sun's light, which they reflect.

COR. 2. From what has been said, we may likewise understand why comets are so frequently seen in that hemisphere in which the sun is, and so seldom in the other. If they were visible in the regions far above Saturn, they would appear more frequently in the parts opposite to the sun; for such as were in those parts would be nearer to the earth, whereas the presence of the sun must obscure and hide those that appear in the hemisphere in which he is. Yet, looking over the history of comets, I find that four or five times more have been seen in the hemisphere towards the sun than in the opposite

hemisphere; besides, without doubt, not a few, which have been hid by the light of the sun: for comets descending into our parts neither emit tails, nor are so well illuminated by the sun, as to discover themselves to our naked eyes, until they are come nearer to us than Jupiter. But the far greater part of that spherical space, which is described about the sun with so small an interval, lies on that side of the earth which regards the sun; and the comets in that greater part are commonly more strongly illuminated, as being for the most part nearer to the sun.

COR. 3. Hence also it is evident that the celestial spaces are void of resistance; for though the comets are carried in oblique paths, and sometimes contrary to the course of the planets, yet they move every way with the greatest freedom, and preserve their motions for an exceeding long time, even where contrary to the course of the planets. I am out in my judgment if they are not a sort of planets revolving in orbits returning into themselves with a perpetual motion; for, as to what some writers contend, that they are no other than meteors, led into this opinion by the perpetual changes that happen to their heads, it seems to have no foundation; for the heads of comets are encompassed with huge atmospheres, and the lowermost parts of these atmospheres must be the densest; and therefore it is in the clouds only, not in the bodies of the comets themselves, that these changes are seen. Thus the earth, if it was viewed from the planets, would, without all doubt, shine by the light of its clouds, and the solid body would scarcely appear through the surrounding clouds. Thus also the belts of Jupiter are formed in the clouds of that planet, for they change their position one to another, and the solid body of Jupiter is hardly to be seen through them; and much more must the bodies of comets be hid under their atmospheres, which are both deeper and thicker.

PROPOSITION XL. THEOREM XX.

That the comets move in some of the conic sections, having their foci in the centre of the sun; and by radii drawn to the sun describe areas proportional to the times.

This proposition appears from cor. 1, prop. 13, book 1, compared with prop. 8, 12, and 13, book 3.

COR. 1. Hence if comets are revolved in orbits returning into themselves, those orbits will be ellipses; and their periodic times be to the periodic times of the planets in the sesquuplicate proportion of their principal axes. And therefore the comets, which for the most part of their course are higher than the planets, and upon that account describe orbits with greater axes, will require a longer time to finish their revolutions. Thus if the axis of a comet's orbit was four times greater than the axis of the orbit of Saturn, the time of the revolution of the comet would be to the time of the revolution of Saturn, that is, to 30 years, as $4 \sqrt{4}$ (or 8) to 1, and would therefore be 240 years.

COR. 2. But their orbits will be so near to parabolas, that parabolas may be used for them without sensible error.

COR. 3. And, therefore, by cor. 7, prop. 16, book 1, the velocity of every comet will always be to the velocity of any planet, supposed to be revolved at the same distance in a circle about the sun, nearly in the subduplicate proportion of double the distance of the planet from the centre of the sun to the distance of the comet from the sun's centre, very nearly. Let us suppose the radius of the *orbis magnus*, or the greatest semidiameter of the ellipse which the earth describes, to consist of 100000000 parts; and then the earth by its mean diurnal motion will describe 1720212 of those parts, and $71675\frac{1}{2}$ by its horary motion. And therefore the comet, at the same mean distance of the earth from the sun, with a velocity which is to the velocity of the earth as $\sqrt{2}$ to 1, would by its diurnal motion describe 2432747 parts, and $101364\frac{1}{2}$ parts by its horary motion. But at greater or less distances both the diurnal and horary motion will be to this diurnal and horary motion in the reciprocal subduplicate proportion of the distances, and is therefore given.

COR. 4. Wherefore if the *latus rectum* of the parabola is quadruple of the radius of the *orbis magnus*, and the square of that radius is supposed to consist of 100000000 parts, the

area which the comet will daily describe by a radius drawn to the sun will be $1216373\frac{1}{2}$ parts, and the horary area will be $50682\frac{1}{4}$ parts. But, if the *latus rectum* is greater or less in any proportion, the diurnal and horary area will be less or greater in the subduplicate of the same proportion reciprocally.

LEMMA V.

To find a curve line of the parabolic kind which shall pass through any given number of points. (Pl. 15, Fig. 3.)

Let those points be A, B, C, D, E, F, &c. and from the same to any right line HN, given in position, let fall as many perpendiculars AH, BI, CK, DL, EM, FN, &c.

CASE 1. If HI, IK, KL, &c. the intervals of the points H, I, K, L, M, N, &c. are equal, take $b, 2b, 3b, 4b, 5b, \&c.$ the first differences of the perpendiculars AH, BI, CK, &c.; their second differences $c, 2c, 3c, 4c, \&c.$; their third, $d, 2d, 3d, \&c.$ that is to say, so as $AH - BI$ may be $= b, BI - CK = 2b, CK - DL = 3b, DL - EM = 4b, -EM + FN = 5b, \&c.$; then $b - 2b = c, \&c.$ and so on to the last difference, which is here f . Then, erecting any perpendicular RS, which may be considered as an ordinate of the curve required, in order to find the length of this ordinate, suppose the intervals HI, IK, KL, LM, &c. to be units, and let $AH = a, -HS = p, \frac{1}{2}p$ into $-IS = q, \frac{1}{3}q$ into $+SK = r, \frac{1}{4}r$ into $+SL = s, \frac{1}{5}s$ into $+SM = t$; proceeding, to wit, to ME, the last perpendicular but one, and prefixing negative signs before the terms HS, IS, &c. which lie from S towards A; and affirmative signs before the terms SK, SL, &c. which lie on the other side of the point S; and observing well the signs, RS will be $= a + bp + cq + dr + es + ft, + \&c.$

CASE 2. But if HI, IK, &c. the intervals of the points H, I, K, L, &c. are unequal, take $b, 2b, 3b, 4b, 5b, \&c.$ the first differences of the perpendiculars AH, BI, CK, &c. divided by the intervals between those perpendiculars; $c, 2c, 3c, 4c, \&c.$ their second differences, divided by the intervals between every two; $d, 2d, 3d, \&c.$ their third differences, divided by the intervals between every three; $e, 2e, \&c.$ their fourth differences, divided by the intervals between every

four; and so forth; that is, in such manner, that b may be $= \frac{AH - BI}{HI}$, $2b = \frac{BI - CK}{IK}$, $3b = \frac{CK - DL}{KL}$, &c. then $c = \frac{b - 2b}{HK}$, $2c = \frac{2b - 3b}{IL}$, $3c = \frac{3b - 4b}{KM}$, &c. then $d = \frac{c - 2c}{HL}$, $2d = \frac{2c - 3c}{IM}$, &c. And those differences being found, let

AH be $= a$, $-HS = p$, p into $-IS = q$, q into $+SK = r$, r into $+SL = s$, s into $+SM = t$; proceeding, to wit, to ME , the last perpendicular but one; and the ordinate RS will be $= a + bp + cq + dr + es + ft + \&c.$

COR. Hence the areas of all curves may be nearly found; for if some number of points of the curve to be squared are found, and a parabola be supposed to be drawn through those points, the area of this parabola will be nearly the same with the area of the curvilinear figure proposed to be squared: but the parabola can be always squared geometrically by methods commonly known.

LEMMA VI.

Certain observed places of a comet being given, to find the place of the same to any intermediate given time.

Let HI, IK, KL, LM (in the preceding Fig.), represent the times between the observations; HA, IB, KC, LD, ME , five observed longitudes of the comet; and HS the given time between the first observation and the longitude required. Then if a regular curve $ABCDE$ is supposed to be drawn through the points A, B, C, D, E , and the ordinate RS is found out by the preceding lemma, RS will be the longitude required.

After the same method, from five observed latitudes, we may find the latitude to a given time.

If the differences of the observed longitudes are small, suppose of 4 or 5 degrees, three or four observations will be sufficient to find a new longitude and latitude; but if the differences are greater, as of 10 or 20 degrees, five observations ought to be used.

LEMMA VII.

Through a given point P (Pl. 15, Fig. 4) to draw a right line BC , whose parts PB, PC , cut off by two right lines AB, AC ,

given in position, may be one to the other in a given proportion.

From the given point P suppose any right line PD to be drawn to either of the right lines given, as AB ; and produce the same towards AC , the other given right line, as far as E , so as PE may be to PD in the given proportion. Let EC be parallel to AD . Draw CPB , and PC will be to PB as PE to PD . Q.E.F.

LEMMA VIII.

Let ABC (Pl. 16, Fig. 1) be a parabola, having its focus in S .

By the chord AC bisected in I cut off the segment $ABCI$, whose diameter is $I\mu$ and vertex μ . In $I\mu$ produced take μO equal to one half of $I\mu$. Join OS , and produce it to ξ , so as $S\xi$ may be equal to $2SO$. Now, supposing a comet to revolve in the arc CBA , draw ξB cutting AC in E ; I say, the point E will cut off from the chord AC the segment AE , nearly proportional to the time.

For if we join EO , cutting the parabolic arc ABC in Y , and draw μX touching the same arc in the vertex μ , and meeting EO in X , the curvilinear area $AEX\mu A$ will be to the curvilinear area $ACY\mu A$ as AE to AC ; and, therefore, since the triangle ASE is to the triangle ASC in the same proportion, the whole area $ASEX\mu A$ will be to the whole area $ASCY\mu A$ as AE to AC . But, because ξO is to SO as 3 to 1, and EO to XO in the same proportion, SX will be parallel to EB ; and, therefore, joining BX , the triangle SEB will be equal to the triangle XEB . Wherefore if to the area $ASEX\mu A$ we add the triangle EXB , and from the sum subduct the triangle SEB , there will remain the area $ASBX\mu A$, equal to the area $ASEX\mu A$, and therefore in proportion to the area $ASCY\mu A$ as AE to AC . But the area $ASBY\mu A$ is nearly equal to the area $ASBX\mu A$; and this area $ASBY\mu A$ is to the area $ASCY\mu A$ as the time of description of the arc AB to the time of description of the whole arc AC ; and, therefore, AE is to AC nearly in the proportion of the times. Q.E.D.

COR. When the point B falls upon the vertex μ of the parabola, AE is to AC accurately in the proportion of the times.

If we join $\mu\xi$ cutting AC in δ , and in it take ξn in proportion to μB as 27MI to 16M μ , and draw Bn, this Bn will cut the chord AC, in the proportion of the times, more accurately than before; but the point n is to be taken beyond or on this side the point ξ , according as the point B is more or less distant from the principal vertex of the parabola than the point μ .

LEMMA IX.

The right lines I μ and μM , and the length $\frac{AIC}{4S\mu}$, are equal among themselves.

For 4S μ is the latus rectum of the parabola belonging to the vertex μ .

LEMMA X.

Produce S μ to N and P (Pl. 16, Fig. 2), so as μN may be one third of μI , and SP may be to SN as SN to S μ ; and in the time that a comet would describe the arc A μ C, if it was supposed to move always forwards with the velocity which it hath in a height equal to SP, it would describe a length equal to the chord AC.

For if the comet with the velocity which it hath in μ was in the said time supposed to move uniformly forwards in the right line which touches the parabola in μ , the area which it would describe by a radius drawn to the point S would be equal to the parabolic area ACS μ A; and therefore the space contained under the length described in the tangent and the length S μ would be to the space contained under the lengths AC and SM as the area ASC μ A to the triangle ASC, that is, as SN to SM. Wherefore AC is to the length described in the tangent as S μ to SN. But since the velocity of the comet in the height SP (by cor. 6, prop. 16, book 1) is to the velocity of the same in the height S μ in the reciprocal subduplicate proportion of SP to S μ , that is, in the proportion of S μ to SN, the length described with this velocity will be to the length in the same time described in the tangent as S μ to SN. Wherefore since AC, and the length described with this new velocity, are in the same proportion to the length described in the tangent, they must be equal betwixt themselves. Q.E.D.

COR. Therefore a comet, with that velocity which it hath in the height $\frac{2}{3}S\mu + I\mu$, would in the same time describe the chord AC nearly.

LEMMA XI.

If a comet void of all motion was let fall from the height SN, or $S\mu + \frac{1}{3}I\mu$, towards the sun, and was still impelled to the sun by the same force uniformly continued by which it was impelled at first, the same, in one half of that time in which it might describe the arc AC in its own orbit, would in descending describe a space equal to the length $I\mu$.

For in the same time that the comet would require to describe the parabolic arc AC, it would (by the last lemma), with that velocity which it hath in the height SP, describe the chord AC; and, therefore (by cor. 7, prop. 16, book 1), if it was in the same time supposed to revolve by the force of its own gravity in a circle whose semidiameter was SP, it would describe an arc of that circle, the length of which would be to the chord of the parabolic arc AC in the subduplicate proportion of 1 to 2. Wherefore if with that weight, which in the height SP it hath towards the sun, it should fall from that height towards the sun, it would (by cor. 9, prop. 4, book 1) in half the said time describe a space equal to the square of half the said chord applied to quadruple the height SP, that is, it would describe the space $\frac{AI^2}{4SP}$. But since the weight of

the comet towards the sun in the height SN is to the weight of the same towards the sun in the height SP as SP to $S\mu$, the comet, by the weight which it hath in the height SN, in falling from that height towards the sun, would in the same time describe the space $\frac{AI^2}{4S\mu}$; that is, a space equal to the length

$I\mu$ or μM . Q.E.D.

PROPOSITION XLI. PROBLEM XXI.

From three observations given, to determine the orbit of a comet moving in a parabola.

This being a problem of very great difficulty, I tried many methods of resolving it; and several of those problems, the

composition whereof I have given in the first book, tended to this purpose. But afterwards I contrived the following solution, which is something more simple.

Select three observations distant one from another by intervals of time nearly equal; but let that interval of time in which the comet moves more slowly be somewhat greater than the other; so, to wit, that the difference of the times may be to the sum of the times as the sum of the times to about 600 days; or that the point E (Pl. 16, Fig. 1) may fall upon M nearly, and may err therefrom rather towards I than towards A. If such direct observations are not at hand, a new place of the comet must be found, by lem. 6.

Let S (Pl. 16, Fig. 3) represent the sun; T, t, τ , three places of the earth in the *orbis magnus*; TA, tB, τ C, three observed longitudes of the comet; V the time between the first observation and the second; W the time between the second and the third; X the length which in the whole time $V + W$ the comet might describe with that velocity which it hath in the mean distance of the earth from the sun, which length is to be found by cor. 3, prop. 40, book 3; and tV a perpendicular upon the chord T τ . In the mean observed longitude tB take at pleasure the point B, for the place of the comet in the plane of the ecliptic; and from thence, towards the sun S, draw the line BE, which may be to the perpendicular tV as the content under SB and St² to the cube of the hypotenuse of the right angled triangle, whose sides are SB, and the tangent of the latitude of the comet in the second observation to the radius tB. And through the point E (by lemma 7) draw the right line AEC, whose parts AE and EC, terminating in the right lines TA and τ C, may be one to the other as the times V and W: then A and C will be nearly the places of the comet in the plane of the ecliptic in the first and third observations, if B was its place rightly assumed in the second.

Upon AC, bisected in I, erect the perpendicular Ii. Through B draw the obscure line Bi parallel to AC. Join the obscure line Si, cutting AC in λ , and complete the parallelogram iI λ μ . Take I σ equal to 3I λ ; and through the sun S draw the obscure

line $\sigma\xi$ equal to $3S\sigma + 3i\lambda$. Then, cancelling the letters A, E, C, I, from the point B towards the point ξ , draw the new obscure line BE, which may be to the former BE in the duplicate proportion of the distance BS to the quantity $S\mu + \frac{1}{3}i\lambda$. And through the point E draw again the right line AEC by the same rule as before, that is, so as its parts AE and EC may be one to the other as the times V and W between the observations. Thus A and C will be the places of the comet more accurately.

Upon AC, bisected in I, erect the perpendiculars AM, CN, IO, of which AM and CN may be the tangents of the latitudes in the first and third observations, to the radii TA and τC . Join MN, cutting IO in O. Draw the rectangular parallelogram $iI\lambda\mu$, as before. In IA produced take ID equal to $S\mu + \frac{2}{3}i\lambda$. Then in MN, towards N, take MP, which may be to the above found length X in the subduplicate proportion of the mean distance of the earth from the sun (or of the semi-diameter of the *orbis magnus*) to the distance OD. If the point P falls upon the point N; A, B, and C, will be three places of the comet, through which its orbit is to be described in the plane of the ecliptic. But if the point P falls not upon the point N, in the right line AC take CG equal to NP, so as the points G and P may lie on the same side of the line NC.

By the same method as the points E, A, C, G, were found from the assumed point B, from other points b and β assumed at pleasure, find out the new points e, a, c, g; and $\epsilon, \alpha, \nu, \gamma$. Then through G, g, and γ , draw the circumference of a circle Gg γ , cutting the right line τC in Z; and Z will be one place of the comet in the plane of the ecliptic. And in AC, ac, $\alpha\kappa$, taking AF, af, $\alpha\phi$, equal respectively to CG, cg, $\nu\gamma$; through the points F, f, and ϕ , draw the circumference of a circle Ff ϕ , cutting the right line AT in X; and the point X will be another place of the comet in the plane of the ecliptic. And at the points X and Z, erecting the tangents of the latitudes of the comet to the radii TX and τZ , two places of the comet in its own orbit will be determined. Lastly, if (by

prop. 19, book 1) to the focus S a parabola is described passing through those two places, this parabola will be the orbit of the comet. Q.E.I.

The demonstration of this construction follows from the preceding lemmas, because the right line AC is cut in E in the proportion of the times, by lem. 7, as it ought to be, by lem. 8; and BE, by lem. 11, is a portion of the right line BS or B ξ in the plane of the ecliptic, intercepted between the arc ABC and the chord AEC; and MP (by cor. lem. 10) is the length of the chord of that arc, which the comet should describe in its proper orbit between the first and third observation, and therefore is equal to MN, providing B is a true place of the comet in the plane of the ecliptic.

But it will be convenient to assume the points B, b, β , not at random, but nearly true. If the angle AQt, at which the projection of the orbit in the plane of the ecliptic cuts the right line tB, is rudely known, at that angle with Bt draw the obscure line AC, which may be to $\frac{4}{3}T\tau$ in the subduplicate proportion of SQ to St; and, drawing the right line SEB so as its part EB may be equal to the length Vt, the point B will be determined, which we are to use for the first time. Then, cancelling the right line AC, and drawing anew AC according to the preceding construction, and, moreover, finding the length MP, in tB take the point b, by this rule, that, if TA and τC intersect each other in Y, the distance Yb may be to the distance YB in a proportion compounded of the proportion of MP to MN, and the subduplicate proportion of SB to Sb. And by the same method you may find the third point β , if you please to repeat the operation the third time; but if this method is followed, two operations generally will be sufficient; for if the distance Bb happens to be very small, after the points F, f, and G, g, are found, draw the right lines Ff and Gg, and they will cut TA and τC in the points required, X and Z.

EXAMPLE.

Let the comet of the year 1680 be proposed. The following table shews the motion thereof, as observed by *Flamsted*,

and calculated afterwards by him from his observations, and corrected by Dr. *Halley* from the same observations.

	Time.		Sun's Longitude.	Comet's			
	Appar.	True		Longitude.		Lat. N.	
	h. "	h. "		o' "	o' "	o' "	o' "
1680, Dec. 12	4 46	4 46 0	♊ 1 51 23	♊ 6 32 39	♊ 8 28 0		
	21 6 32½	6 36 59	11 06 44	♊ 5 08 12	21 42 13		
	24 6 12	6 17 52	14 09 26	18 49 23	25 23 5		
	26 5 14	5 20 44	16 09 22	28 24 13	27 00 52		
	29 7 55	8 03 02	19 19 43	♋ 13 10 41	28 09 58		
	30 8 02	8 10 26	20 21 09	17 38 20	28 11 53		
1681, Jan. 5	5 51	6 01 38	26 22 18	♋ 8 48 53	26 15 7		
	9 6 49	7 00 53	♋ 0 29 02	18 44 04	24 11 56		
	10 5 54	6 06 10	1 27 43	20 40 50	23 43 52		
	13 6 56	7 08 55	4 33 20	25 59 48	22 17 28		
	25 7 44	7 58 42	16 45 36	♌ 9 35 0	17 56 30		
	30 8 07	8 21 53	21 49 58	13 19 51	16 42 18		
Feb. 2	6 20	6 34 51	24 46 59	15 13 53	16 04 1		
	5 6 50	7 04 41	27 49 51	16 59 06	15 27 3		

To these you may add some observations of mine.

	Ap. Time.	Comet's			
		Longitude.		Lat. N.	
		o' "	o' "	o' "	o' "
1681, Feb. 25	8 30	♌ 26 18 35	12 46 46		
	27	27 04 20	22 36 12		
Mar. 1	11 0	27 52 42	12 23 40		
	2	23 12 48	12 19 38		
	5 11 30	29 18 0	12 03 16		
	7 9 30	♍ 0 4 0	11 57 0		
	9 8 30	0 43 4	11 45 52		

These observations were made by a telescope of 7 feet, with a micrometer and threads placed in the focus of the telescope; by which instruments we determined the positions both of the fixed stars among themselves, and of the comet in respect of the fixed stars. Let A (Pl. 17) represent the star of the fourth magnitude in the left heel of *Perseus* (*Bayer's* α), B the following star of the third magnitude in the left foot (*Bayer's* ζ), C a star of the sixth magnitude (*Bayer's* η) in the heel of the same foot, and D, E, F, G, H, I, K, L, M, N, O, Z, α , β , γ , δ , other smaller stars in the same foot; and let p, P, Q, R, S, T, V, X, represent the

places of the comet in the observations above set down; and, reckoning the distance AB of $80\frac{7}{12}$ parts, AC was $52\frac{1}{4}$ of those parts; BC, $58\frac{5}{8}$; AD, $57\frac{5}{12}$; BD, $82\frac{6}{11}$; CD, $23\frac{2}{3}$; AE, $29\frac{4}{7}$; CE, $57\frac{1}{2}$; DE, $49\frac{11}{12}$; AI, $27\frac{7}{12}$; BI, $52\frac{1}{8}$; CI, $36\frac{7}{12}$; DI, $53\frac{5}{11}$; AK, $38\frac{2}{3}$; BK, 43; CK, $31\frac{5}{9}$; FK, 29; FB, 23; FC, $36\frac{1}{4}$; AH, $18\frac{6}{7}$; DH, $50\frac{7}{8}$; BN, $46\frac{5}{12}$; CN, $31\frac{1}{3}$; BL, $45\frac{5}{12}$; NL, $31\frac{5}{7}$. HO was to HI as 7 to 6, and, produced, did pass between the stars D and E, so as the distance of the star D from this right line was $\frac{1}{6}$ CD. LM was to LN as 2 to 9, and, produced, did pass through the star H. Thus were the positions of the fixed stars determined in respect of one another.

Mr. Pound has since observed, a second time, the positions of these fixed stars amongst themselves, and collected their longitudes and latitudes according to the following table.

The fixed stars	Their Longitudes.				Latitude North.			The fixed stars	Their Longitudes.				Latitude North.		
	°	'	"		°	'	"		°	'	"	°	'	"	
A	8	26	41	50	12	8	36	L	8	29	33	34	12	7	48
B		28	40	23	11	17	54	M		29	18	54	12	7	20
C		27	58	30	12	40	25	N		28	48	29	12	31	9
E		26	27	17	12	52	7	Z		29	44	48	11	57	13
F		28	28	37	11	52	22	α		29	52	3	11	55	48
G		26	56	8	12	4	58	β	Π	0	8	23	11	48	56
H		27	11	45	12	2	1	γ		0	40	10	11	55	18
I		27	25	2	11	53	11	δ		1	3	20	11	30	42
K		27	42	7	11	53	26								

The positions of the comet to these fixed stars were observed to be as follow :

Friday, *February* 25, O.S. at $8\frac{1}{2}$ h. P.M. the distance of the comet in p from the star E was less than $\frac{3}{13}$ AE, and greater than $\frac{1}{5}$ AE, and therefore nearly equal to $\frac{3}{14}$ AE; and the angle ApE was a little obtuse, but almost right. For from A, letting fall a perpendicular on pE, the distance of the comet from that perpendicular was $\frac{1}{5}$ pE.

The same night, at $9\frac{1}{2}$ h. the distance of the comet in P from the star E was greater than $\frac{1}{41\frac{1}{2}}$ AE, and less than $\frac{1}{54}$ AE, and therefore nearly equal to $\frac{10}{47\frac{1}{8}}$ of AE, or $\frac{8}{29}$ AE. But

the distance of the comet from the perpendicular let fall from the star A upon the right line PE was $\frac{4}{5}$ PE.

Sunday, 27 *February*, 8 $\frac{1}{4}$ ^h. P.M., the distance of the comet in Q from the star O was equal to the distance of the stars O and H; and the right line QO produced passed between the stars K and B. I could not, by reason of intervening clouds, determine the position of the star to greater accuracy.

Tuesday, 1 *March*, 11^h. P.M., the comet in R lay exactly in a line between the stars K and C, so as the part CR of the right line CRK was a little greater than $\frac{1}{3}$ CK, and a little less than $\frac{1}{3}$ CK + $\frac{1}{8}$ CR, and therefore $= \frac{1}{3}$ CK + $\frac{1}{15}$ CR, or $\frac{16}{45}$ CK.

Wednesday, 2 *March*, 8^h. P.M., the distance of the comet in S from the star C was nearly $\frac{4}{3}$ FC; the distance of the star F from the right line CS produced was $\frac{1}{24}$ FC; and the distance of the star B from the same right line was five times greater than the distance of the star F; and the right line NS produced passed between the stars H and I five or six times nearer to the star H than to the star I.

Saturday, 5 *March*, 11 $\frac{1}{2}$ ^h. P.M., when the comet was in T, the right line MT was equal to $\frac{1}{2}$ ML, and the right line LT produced passed between B and F four or five times nearer to F than to B, cutting off from BF a fifth or sixth part thereof towards F; and MT produced passed on the outside of the space BF towards the star B four times nearer to the star B than to the star F. M was a very small star, scarcely to be seen by the telescope; but the star L was greater, and of about the eighth magnitude.

Monday, 7 *March*, 9 $\frac{1}{2}$ ^h. P.M., the comet being in V, the right line V α produced did pass between B and F, cutting off, from BF towards F, $\frac{1}{10}$ of BF, and was to the right line V β as 5 to 4. And the distance of the comet from the right line $\alpha\beta$ was $\frac{1}{2}$ V β .

Wednesday, 9 *March*, 8 $\frac{1}{2}$ ^h. P.M., the comet being in X, the right line γX was equal to $\frac{1}{4}\gamma\delta$; and the perpendicular let fall from the star δ upon the right line γX was $\frac{2}{5}$ of $\gamma\delta$.

The same night, at 12^h. the comet being in Y, the right line γY was equal to $\frac{1}{3}$ of $\gamma\delta$, or a little less, as perhaps $\frac{5}{16}$ of $\gamma\delta$; and a perpendicular let fall from the star δ on the right

line γY was equal to about $\frac{1}{8}$ or $\frac{1}{7}\gamma\delta$. But the comet being then extremely near the horizon, was scarcely discernible, and therefore its place could not be determined with that certainty as in the foregoing observations.

From these observations, by constructions of figures and calculations, I deduced the longitudes and latitudes of the comet; and Mr. *Pound*, by correcting the places of the fixed stars, hath determined more correctly the places of the comet, which correct places are set down above. Though my micrometer was none of the best, yet the errors in longitude and latitude (as derived from my observations) scarcely exceed one minute. The comet (according to my observations), about the end of its motion, began to decline sensibly towards the north, from the parallel which it described about the end of *February*.

Now, in order to determine the orbit of the comet out of the observations above described, I selected those three which *Flamsted* made, *Dec.* 21, *Jan.* 5, and *Jan.* 25; from which I found *St* of 9842,1 parts, and *Vt* of 455, such as the semidiameter of the *orbis magnus* contains 10000. Then for the first observation, assuming *tB* of 5657 of those parts, I found *SB* 9747, *BE* for the first time 412, *Sμ* 9503, *ia* 413, *BE* for the second time 421, *OD* 10186, *X* 8528,4, *PM* 8450, *MN* 8475, *NP* 25; from whence, by the second operation, I collected the distance *tb* 5640; and by this operation I at last deduced the distances *TX* 4775 and *τZ* 11322. From which, limiting the orbit, I found its descending node in ϖ , and ascending node in ϖ $1^{\circ} 53'$; the inclination of its plane to the plane of the ecliptic $61^{\circ} 20\frac{1}{3}'$; the vertex thereof (or the perihelion of the comet) distant from the node $8^{\circ} 38'$, and in $\uparrow$ $27^{\circ} 43'$, with latitude $7^{\circ} 34'$ south; its *latus rectum* 236,8; and the diurnal area described by a radius drawn to the sun 93585, supposing the square of the semidiameter of the *orbis magnus* 100000000; that the comet in this orbit moved directly according to the order of the signs, and on *Dec.* 8^d. 00^h. 04^m P.M. was in the vertex or perihelion of its orbit. All which I determined by scale and compass, and the chords of angles, taken from the table of natural sines, in a pretty large figure,

in which, to wit, the radius of the *orbis magnus* (consisting of 10000 parts) was equal to $16\frac{1}{3}$ inches of an *English* foot.

Lastly, in order to discover whether the comet did truly move in the orbit so determined, I investigated its places in this orbit partly by arithmetical operations, and partly by scale and compass, to the times of some of the observations, as may be seen in the following table.

The COMET'S							
	Dist. from sun.	Longitude computed.	Latitude com- puted.	Longitude observed.	Latitude observed.	Dif. Long.	Dif. Lat.
Dec. 12	2792	φ $6^{\circ} 32'$	$8^{\circ} 18\frac{1}{2}'$	φ $6^{\circ} 31\frac{1}{2}'$	$8^{\circ} 26'$	+ 1	— $7\frac{1}{2}'$
29	8403	$\propto$ $13^{\circ} 13\frac{2}{3}'$	$28^{\circ} 00'$	$\propto$ $13^{\circ} 11\frac{3}{4}'$	$28^{\circ} 10\frac{1}{12}'$	+ 2	— $10\frac{1}{12}'$
Febr. 5	16669	δ $17^{\circ} 00'$	$15^{\circ} 29\frac{2}{3}'$	δ $16^{\circ} 59\frac{7}{8}'$	$15^{\circ} 27\frac{2}{3}'$	+ 0	+ $2\frac{1}{2}'$
Mar. 5	21737	$29^{\circ} 19\frac{3}{4}'$	$12^{\circ} 4'$	$29^{\circ} 20\frac{6}{7}'$	$12^{\circ} 3\frac{1}{2}'$	— 1	+ $\frac{1}{2}'$

But afterwards Dr. *Halley* did determine the orbit to a greater accuracy by an arithmetical calculus than could be done by linear descriptions; and, retaining the place of the nodes in φ and φ $1^{\circ} 53'$, and the inclination of the plane of the orbit to the ecliptic $61^{\circ} 20\frac{1}{3}'$, as well as the time of the comet's being in perihelio, Dec. 8^d. 00^h. 04', he found the distance of the perihelion from the ascending node measured in the comet's orbit $9^{\circ} 20'$, and the *latus rectum* of the parabola 2430 parts, supposing the mean distance of the sun from the earth to be 100000 parts: and from these data, by an accurate arithmetical calculus, he computed the places of the comet to the times of the observations as follows.

True time.				The COMET'S				Errors in	
d.	h.	m.	s.	Dist. from the sun.	Longitude computed.	Latitude computed.	Bor.	Long.	Lat.
Dec.	12	4	46	28028	6	29	25	3	5
	21	6	37	61076	5	6	30	1	42
	24	6	18	70008	18	48	20	1	3
	26	5	20	75576	28	22	45	1	28
	29	8	3	84021	13	12	40	1	59
Jan.	30	8	10	86661	17	40	5	1	45
	5	6	1	101440	8	49	49	0	56
	9	7	0	110959	18	44	36	0	32
	10	6	6	113162	20	41	0	0	10
	13	7	9	120000	26	0	21	0	33
Febr.	25	7	59	145370	9	33	40	1	20
	30	8	22	155303	13	17	41	2	10
	2	6	35	160951	15	11	11	2	42
	5	7	4	166686	16	58	55	0	41
	25	8	41	202570	26	15	46	2	49
Mar.	5	11	39	216205	29	18	35	0	35

This comet also appeared in the *November* before, and at *Coburg*, in *Saxony*, was observed by Mr. *Gottfried Kirch*, on the 4th of that month, on the 6th and 11th O. S; from its positions to the nearest fixed stars observed with sufficient accuracy, sometimes with a two feet and sometimes with a ten feet telescope; from the difference of longitudes of *Coburg* and *London*, 11° ; and from the places of the fixed stars observed by Mr. *Pound*, Dr. *Halley* has determined the places of the comet as follows.

3 Nov., 17^h. 2', apparent time at *London*, the comet was in Ω 29 deg. 51', with 1 deg. 17' 45" latitude north.

5 November, 15^h. 58', the comet was in π 3° 23', with 1° 6' north lat.

10 November, 16^h 31', the comet was equally distant from two stars in Ω , which are σ and τ in *Bayer*; but it had not quite touched the right line that joins them, but was very little distant from it. In *Flamsteed's* catalogue this star σ was then in π 14° 15', with 1 deg. 41' lat. north nearly, and τ in π 17° 3½' with 0 deg. 34' lat. south; and the middle point between those stars was π 15° 39¼', with 0° 33½' lat. north. Let the distance of the comet from that right line be about 10' or 12'; and the difference of the longitude of the comet and that middle point will be 7'; and the difference of the latitude nearly 7½'; and thence it follows that the comet was in π 15° 32', with about 26' lat. north.

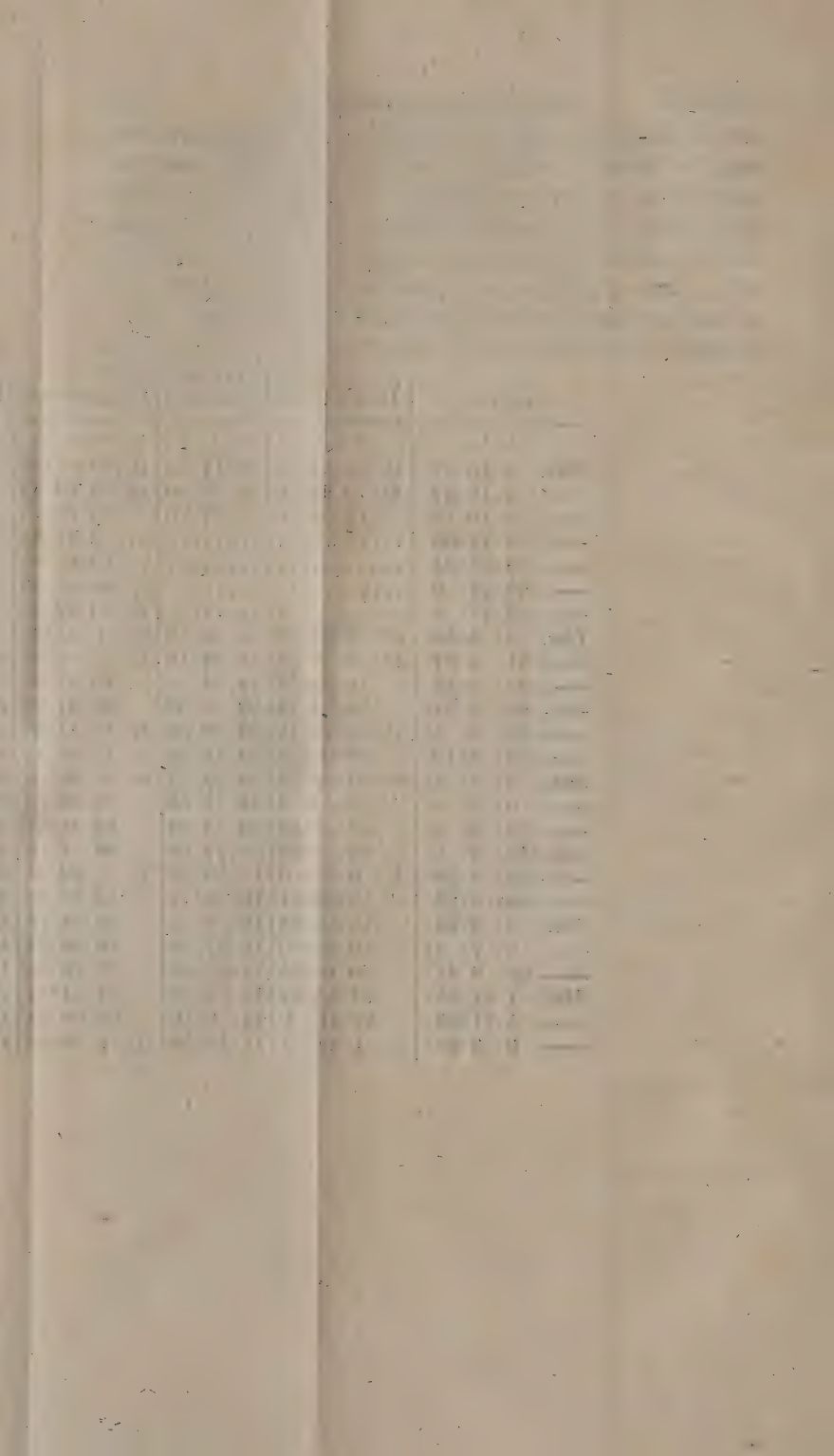
The first observation from the position of the comet with respect to certain small fixed stars had all the exactness that could be desired; the second also was accurate enough. In the third observation, which was the least accurate, there might be an error of 6 or 7 minutes, but hardly greater. The longitude of the comet, as found in the first and most accurate observation, being computed in the aforesaid parabolic orbit, comes out Ω 29° 30' 22", its latitude north 1° 25' 7", and its distance from the sun 115546.

Moreover, Dr. *Halley*, observing that a remarkable comet had appeared four times at equal intervals of 575 years (that is, in the month of *September* after *Julius Cæsar* was killed; *An. Chr.* 531, in the consulate of *Lampadius* and *Orestes*; *An. Chr.* 1106, in the month of *February*; and at the end of the year 1680; and that with a long and remarkable tail, except when it was seen after *Cæsar's* death, at which time, by reason of the inconvenient situation of the earth, the tail was not so conspicuous), set himself to find out an elliptic orbit whose greater axis should be 1382957 parts, the mean distance of the earth from the sun containing 10000 such; in which orbit a comet might revolve in 575 years; and, placing the ascending node in ϖ 2° 2', the inclination of the plane of the

orbit to the plane of the ecliptic in an angle of $61^{\circ} 6' 48''$, the perihelion of the comet in this plane in $\uparrow 22^{\circ} 44' 25''$, the equal time of the perihelion *December 7^a. 23^b. 9'*, the distance of the perihelion from the ascending node in the plane of the ecliptic $9^{\circ} 17' 35''$, and its conjugate axis 18481,2, he computed the motions of the comet in this elliptic orbit. The places of the comet, as deduced from the observations, and as arising from computation made in this orbit, may be seen in the following table.

[To face Page 278.]

True time.	Longitude obs.			Latitude Nor. obs.			Longitude comp			Latitude comp			Errors in							
	d.	h.	'	°	'	"	°	'	"	°	'	"	Longitude	Latitude.						
Nov.	3	16	47	Ω	29	51	0	1	17	45	Ω	29	51	22	1	17	32	N	+ 0 22	- 0 13
	5	15	37	♊	3	23	0	1	6	0	♊	3	24	32	1	6	9		+ 1 32	+ 0 9
	10	16	18		15	32	0	0	27	0		15	33	2	0	25	7		+ 1 2	- 1 53
	16	17	00								♋	8	16	45	0	53	7	S		
	18	21	34								..	18	52	15	1	26	54			
	20	17	0								..	28	10	36	1	53	35			
	23	17	5								♌	13	22	42	2	29	0			
Dec.	12	4	46	♍	6	32	30	8	28	0	♍	6	31	20	8	29	6	N	- 1 10	+ 1 6
	21	6	37	♎	5	8	12	21	42	13	♎	5	6	14	21	44	42		- 1 58	+ 2 29
	24	6	18		18	49	23	25	23	5		18	47	30	25	23	35		- 1 53	+ 0 30
	26	5	21		28	24	13	27	0	52		28	21	42	27	2	1		- 2 31	+ 1 9
	29	8	3	♏	13	10	41	28	9	58	♏	13	11	14	28	10	38		+ 0 33	+ 0 40
	30	8	10		17	38	0	28	11	53		17	38	27	28	11	37		+ 0 7	- 0 16
Jan.	5	6	1 $\frac{1}{2}$	♐	8	48	53	26	15	7	♐	8	48	51	26	14	57		- 0 2	- 0 10
	9	7	1		18	44	4	24	11	56		18	43	51	24	12	17		- 0 13	+ 0 21
	10	6	6		20	40	50	23	43	32		20	40	23	23	43	25		- 0 27	- 0 7
	13	7	9		25	59	48	22	17	28		26	0	8	22	16	32		+ 0 20	- 0 56
	25	7	59	♑	9	35	0	17	56	30	♑	9	34	11	17	56	6		- 0 49	- 0 24
	30	8	22		13	19	51	16	42	18		13	18	28	16	40	5		- 1 23	- 2 13
Feb.	2	6	35		15	13	53	16	4	1		15	11	59	16	2	7		- 1 54	- 1 54
	5	7	4 $\frac{1}{2}$		16	59	6	15	27	3		16	59	17	15	27	0		+ 0 11	- 0 3
	25	8	41		26	18	35	12	46	46		26	16	59	12	45	22		- 1 36	- 1 24
Mar.	1	11	10		27	52	42	12	23	40		27	51	47	12	22	28		- 0 55	- 1 12
	5	11	39		29	18	0	12	3	16		29	20	11	12	2	50		+ 2 11	- 0 26
	9	8	38	♒	0	43	4	11	45	52	♒	0	42	43	11	45	35		- 0 21	- 0 17



The observations of this comet from the beginning to the end agree as perfectly with the motion of the comet in the orbit just now described as the motions of the planets do with the theories from whence they are calculated; and by this agreement plainly evince that it was one and the same comet that appeared all that time, and also that the orbit of that comet is here rightly defined.

In the foregoing table we have omitted the observations of Nov. 16, 18, 20, and 23, as not sufficiently accurate, for at those times several persons had observed the comet. Nov. 17, O. S. *Ponthæus* and his companions, at 6^h. in the morning at *Rome* (that is, 5^h. 10' at *London*) by threads directed to the fixed stars, observed the comet in $\simeq 8^{\circ} 30'$, with latitude $0^{\circ} 40'$ south. Their observations may be seen in a treatise which *Ponthæus* published concerning this comet. *Cellius*, who was present, and communicated his observations in a letter to *Cassini*, saw the comet at the same hour in $\simeq 8^{\circ} 30'$, with latitude $0^{\circ} 30'$ south. It was likewise seen by *Galletius* at the same hour at *Avignon* (that is, at 5^h. 42' morning at *London*) in $\simeq 8^{\circ}$ without latitude. But by the theory the comet was at that time in $\simeq 8^{\circ} 16' 45''$, and its latitude was $0^{\circ} 53' 7''$ south.

18 Nov., at 6^h. 30' in the morning at *Rome* (that is, at 5^h. 40' at *London*), *Ponthæus* observed the comet in $\simeq 13^{\circ} 30'$, with latitude $1^{\circ} 20'$ south; and *Cellius* in $\simeq 13^{\circ} 30'$, with latitude $1^{\circ} 00'$ south. But at 5^h. 30' in the morning at *Avignon*, *Galletius* saw it in $\simeq 13^{\circ} 00'$, with latitude $1^{\circ} 00'$ south. In the University of *La Fleche*, in *France*, at 5^h. in the morning (that is, at 5^h. 9' at *London*), it was seen by *P. Ango*, in the middle between two small stars, one of which is the middle of the three which lie in a right line in the southern hand of *Virgo*, *Bayer's* ψ ; and the other is the utmost of the wing, *Bayer's* θ . Whence the comet was then in $\simeq 12^{\circ} 46'$, with latitude $50'$ south. And I was informed by Dr. *Halley*, that on the same day at *Boston* in *New England*, in the latitude of $42\frac{1}{2}$ deg. at 5^h. in the morning (that is, at 9^h. 44' in the morning at *London*), the comet was seen near $\simeq 14^{\circ}$, with latitude $1^{\circ} 30'$ south.

Nov. 19, at $4\frac{1}{2}^h$. at *Cambridge*, the comet (by the observation of a young man) was distant from *Spica* μ about 2° towards the north west. Now the spike was at that time in α $19^\circ 23' 47''$, with latitude $2^\circ 1' 59''$ south. The same day, at 5^h . in the morning, at *Boston*, in *New England*, the comet was distant from *Spica* μ 1° , with the difference of $40'$ in latitude. The same day, in the island of *Jamaica*, it was about 1° distant from *Spica* μ . The same day. Mr. *Arthur Storer*, at the river *Patuxent*, near *Hunting Creek*, in *Maryland*, in the confines of *Virginia*, in lat. $38\frac{1}{2}^\circ$, at 5 in the morning (that is, at 10^h . at *London*), saw the comet above *Spica* μ , and very nearly joined with it, the distance between them being about $\frac{3}{4}$ of one degree. And from these observations compared, I conclude, that at $9^h. 44'$ at *London* the comet was in α $18^\circ 50'$, with about $1^\circ 25'$ latitude south. Now by the theory the comet was at that time in α $18^\circ 52' 15''$, with $1^\circ 26' 54''$ lat. south.

Nov. 20, *Montenari*, professor of astronomy at *Padua*, at 6^h . in the morning at *Venice* (that is, $5^h. 10'$ at *London*), saw the comet in α 23° , with latitude $1^\circ 30'$ south. The same day, at *Boston*, it was distant from *Spica* μ by about 4° of longitude east, and therefore was in α $23^\circ 24'$ nearly.

Nov. 21, *Ponthæus* and his companions, at $7\frac{1}{4}^h$. in the morning, observed the comet in α $27^\circ 50'$, with latitude $1^\circ 16'$ south; *Cellius*, in α 28° ; *P. Ango* at 5^h . in the morning, in α $27^\circ 45'$; *Montenari* in α $27^\circ 51'$. The same day, in the island of *Jamaica*, it was seen near the beginning of η , and of about the same latitude with *Spica* μ , that is, $2^\circ 2'$. The same day, at 5^h . morning, at *Ballafore*, in the *East Indies* (that is, at $11^h. 20'$ of the night preceding at *London*), the distance of the comet from *Spica* μ was taken $7^\circ 35'$ to the east. It was in a right line between the spike and the balance, and therefore was then in α $26^\circ 58'$, with about $1^\circ 11'$ lat. south; and after $5^h. 40'$ (that is, at 5^h . morning at *London*), it was in α $28^\circ 12'$, with $1^\circ 16'$ lat. south. Now by the theory the comet was then in α $28^\circ 10' 36''$, with $1^\circ 53' 35''$ lat. south.

Nov. 22, the comet was seen by *Montenari* in η $2^\circ 33'$; but at *Boston* in *New England*, it was found in about η 3° , and with almost the same latitude as before, that is, $1^\circ 30'$.

The same day, at 5^h. morning at *Ballafore*, the comet was observed in $m\ 1^{\circ} 50'$; and therefore, at 5^h. morning at *London*, the comet was in $m\ 3^{\circ} 5'$ nearly. The same day, at 6 $\frac{1}{2}$ ^h. in the morning at *London*, Dr. *Hook* observed it in about $m\ 3^{\circ} 30'$, and that in the right line which passeth through *Spica* μ and *Cor Leonis*; not, indeed, exactly, but deviating a little from that line towards the north. *Montenari* likewise observed, that this day, and some days after, a right line drawn from the comet through *Spica* passed by the south side of *Cor Leonis* at a small distance therefrom. The right line through *Cor Leonis* and *Spica* μ did cut the ecliptic in $m\ 3^{\circ} 46'$ at an angle of $2^{\circ} 51'$; and if the comet had been in this line and in $m\ 3^{\circ}$, its latitude would have been $2^{\circ} 26'$: but since *Hook* and *Montenari* agree that the comet was at some small distance from this line towards the north, its latitude must have been something less. On the 20th, by the observation of *Montenari*, its latitude was almost the same with that of *Spica*, that is, about $1^{\circ} 30'$. But by the agreement of *Hook*, *Montenari*, and *Ango*, the latitude was continually increasing, and therefore must now, on the 22d, be sensibly greater than $1^{\circ} 30'$; and, taking a mean between the extreme limits but now stated, $2^{\circ} 26'$ and $1^{\circ} 30'$, the latitude will be about $1^{\circ} 58'$. *Hook* and *Montenari* agree that the tail of the comet was directed towards *Spica* μ , declining a little from that star towards the south according to *Hook*, but towards the north according to *Montenari*; and, therefore, that declination was scarcely sensible; and the tail, lying nearly parallel to the equator, deviated a little from the opposition of the sun towards the north.

Nov. 23, O. S., at 5^h. morning, at *Nuremberg* (that is, at 4 $\frac{1}{2}$ ^h. at *London*), Mr. *Zimmerman* saw the comet in $m\ 8^{\circ} 8'$, with $2^{\circ} 31'$ south lat., its place being collected by taking its distances from fixed stars.

Nov. 24, before sun-rising, the comet was seen by *Montenari* in $m\ 12^{\circ} 52'$ on the north side of the right line through *Cor Leonis* and *Spica* μ , and therefore its latitude was something less than $2^{\circ} 38'$; and since the latitude, as we said, by the concurring observations of *Montenari*, *Ango*, and *Hook*,

was continually increafing, therefore, it was now, on the 24th, fomewhat greater than $1^{\circ} 58'$; and, taking the mean quantity, may be reckoned $2^{\circ} 18'$, without any confiderable error. *Ponthæus* and *Galletius* will have it that the latitude was now decreafing; and *Cellius*, and the obferver in *New England*, that it continued the fame, viz. of about 1° , or $1\frac{1}{2}^{\circ}$. The obfervations of *Ponthæus* and *Cellius* are more rude, efpecially thofe which were made by taking the azimuths and altitudes; as are alfo the obfervations of *Galletius*. Thofe are better which were made by taking the pofition of the comet to the fixed ftars by *Montenari*, *Hook*, *Ango*, and the obferver in *New England*, and fometimes by *Ponthæus* and *Cellius*. The fame day, at 5^h. morning at *Ballafore*, the comet was obferved in $\text{m } 11^{\circ} 45'$; and, therefore, at 5^h. morning at *London*, was in $\text{m } 13^{\circ}$ nearly. And, by the theory, the comet was at that time in $\text{m } 13^{\circ} 22' 42''$.

Nov. 25, before fun-rife, *Montenari* obferved the comet in $\text{m } 17\frac{3}{4}^{\circ}$ nearly; and *Cellius* obferved at the fame time that the comet was in a right line between the bright ftar in the right thigh of *Virgo* and the fouthern fcale of *Libra*; and this right line cuts the comet's way in $\text{m } 18^{\circ} 36'$. And, by the theory, the comet was in $\text{m } 18\frac{1}{3}^{\circ}$ nearly.

From all this it is plain that thefe obfervations agree with the theory, fo far as they agree with one another; and by this agreement it is made clear that it was one and the fame comet that appeared all the time from *Nov. 4* to *March 9*. The path of this comet did twice cut the plane of the ecliptic, and therefore was not a right line. It did cut the ecliptic not in oppofite parts of the heavens, but in the end of *Virgo* and beginning of *Capricorn*, including an arc of about 98° ; and therefore the way of the comet did very much deviate from the path of a great circle; for in the month of *Nov.* it declined at leaft 30° from the ecliptic towards the fouth; and in the month of *Dec.* following it declined 29° from the ecliptic towards the north; the two parts of the orbit in which the comet defcended towards the fun, and afcended again from the fun, declining one from the other by an apparent angle of above 30° , as obferved by *Montenari*. This comet travelled

over 9 signs, to wit, from the last deg. of Ω to the beginning of Π , beside the sign of Ω , through which it passed before it began to be seen; and there is no other theory by which a comet can go over so great a part of the heavens with a regular motion. The motion of this comet was very unequable; for about the 20th of *Nov.* it described about 5° a day. Then its motion being retarded between *Nov.* 26 and *Dec.* 12, to wit, in the space of $15\frac{1}{2}$ days, it described only 40° . But the motion thereof being afterwards accelerated, it described near 5° a day, till its motion began to be again retarded. And the theory which justly corresponds with a motion so unequable, and through so great a part of the heavens, which observes the same laws with the theory of the planets, and which accurately agrees with accurate astronomical observations, cannot be otherwise than true.

And, thinking it would not be improper, I have given (Pl. 18) a true representation of the orbit which this comet described, and of the tail which it emitted in several places, in the annexed figure; protracted in the plane of the trajectory. In this scheme ABC represents the trajectory of the comet, D the sun, DE the axis of the trajectory, DF the line of the nodes, GH the intersection of the sphere of the *orbis magnus* with the plane of the trajectory, I the place of the comet *Nov.* 4, *Ann.* 1680; K the place of the same *Nov.* 11; L the place of the same *Nov.* 19; M its place *Dec.* 12; N its place *Dec.* 21; O its place *Dec.* 29; P its place *Jan.* 5 following; Q its place *Jan.* 25; R its place *Feb.* 5; S its place *Feb.* 25; T its place *March* 5; and V its place *March* 9. In determining the length of the tail, I made the following observations.

Nov. 4 and 6, the tail did not appear; *Nov.* 11, the tail just begun to shew itself, but did not appear above $\frac{1}{2}$ degree long through a 10 feet telescope; *Nov.* 17, the tail was seen by *Ponthæus* more than 15° long; *Nov.* 18, in *New England*, the tail appeared 30° long, and directly opposite to the sun, extending itself to the planet Mars, which was then in μ , $9^{\circ} 54'$; *Nov.* 19, in *Maryland*, the tail was found 15° or 20° long; *Dec.* 10 (by the observation of Mr. *Flamsted*), the

tail passed through the middle of the distance intercepted between the tail of the Serpent of *Ophiuchus* and the star δ in the south wing of *Aquila*, and did terminate near the stars A, ω , b, in *Bayer's* tables. Therefore the end of the tail was in ϖ $19^{\frac{10}{2}}$, with latitude about $34^{\frac{1}{4}}$ ° north; *Dec.* 11, it ascended to the head of *Sagitta* (*Bayer's* α , β), terminating in ϖ $26^{\circ} 43'$, with latitude $38^{\circ} 34'$ north; *Dec.* 12, it passed through the middle of *Sagitta*, nor did it reach much farther; terminating in $\approx$ 4° , with latitude $42^{\frac{1}{2}}$ ° north nearly. But these things are to be understood of the length of the brighter part of the tail; for with a more faint light, observed, too, perhaps, in a serener sky, at *Rome*, *Dec.* 12, $5^h. 40'$, by the observation of *Ponthæus*, the tail arose to 10° above the rump of the Swan, and the side thereof towards the west and towards the north was $45'$ distant from this star. But about that time the tail was 3° broad towards the upper end; and therefore the middle thereof was $2^{\circ} 15'$ distant from that star towards the south, and the upper end was $\times$ in 22° , with latitude 61° north; and thence the tail was about 70° long; *Dec.* 21, it extended almost to *Cassiopeia's* chair, equally distant from β and from *Schedir*, so as its distance from either of the two was equal to the distance of the one from the other, and therefore did terminate in γ 24° , with latitude $47^{\frac{1}{2}}$ °; *Dec.* 29, it reached to a contact with *Scheat* on its left, and exactly filled up the space between the two stars in the northern foot of *Andromeda*, being 54° in length; and therefore terminated in γ 19° , with 35° of latitude; *Jan.* 5, it touched the star π in the breast of *Andromeda* on its right side, and the star μ of the girdle on its left, and, according to our observations, was 40° long; but it was curved, and the convex side thereof lay to the south; and near the head of the comet it made an angle of 4° with the circle which passed through the sun and the comet's head; but towards the other end it was inclined to that circle in an angle of about 10° or 11° ; and the chord of the tail contained with that circle an angle of 8° . *Jan.* 13, the tail terminated between *Alamech* and *Algol*, with a light that was sensible enough; but with a faint light it ended over against the star κ in *Perseus's* side. The distance of the end

of the tail from the circle passing through the sun and the comet was $3^{\circ} 50'$; and the inclination of the chord of the tail to that circle was $8\frac{1}{2}^{\circ}$. *Jan.* 25 and 26, it shone with a faint light to the length of 6° or 7° ; and for a night or two after, when there was a very clear sky, it extended to the length of 12° , or something more, with a light that was very faint and with difficulty to be seen; but the axis thereof was exactly directed to the bright star in the eastern shoulder of *Auriga*, and therefore deviated from the opposition of the sun towards the north by an angle of 10° . Lastly, *Feb.* 10, with a telescope I observed the tail 2° long; for that fainter light which I spoke of did not appear through the glasses. But *Ponthæus* writes, that, on *Feb.* 7, he saw the tail 12° long. *Feb.* 25, the comet was without a tail, and so continued till it disappeared.

Now if any one reflects upon the orbit described, and duly considers the other appearances of this comet, he will be easily satisfied that the bodies of comets are solid, compact, fixed, and durable, like the bodies of the planets; for if they were nothing else but the vapours or exhalations of the earth, of the sun, and other planets, this comet, in its passage by the neighbourhood of the sun, would have been immediately dissipated; for the heat of the sun is as the density of its rays, that is, reciprocally as the square of the distance of the places from the sun. Therefore, since, on *Dec.* 8, when the comet was in its perihelion, the distance thereof from the centre of the sun was to the distance of the earth from the same as about 6 to 1000, the sun's heat on the comet was at that time to the heat of the summer-sun with us as 1000000 to 36, or as 28000 to 1. But the heat of boiling water is about 3 times greater than the heat which dry earth acquires from the summer-sun, as I have tried; and the heat of red-hot iron (if my conjecture is right) is about three or four times greater than the heat of boiling water. And therefore the heat which dry earth on the comet, while in its perihelion, might have conceived from the rays of the sun, was about 2000 times greater than the heat of red-hot iron. But by so fierce a heat, va-

pours and exhalations, and every volatile matter, must have been immediately consumed and diffipated.

This comet, therefore, must have conceived an immense heat from the sun, and retained that heat for an exceeding long time; for a globe of iron of an inch in diameter, exposed red-hot to the open air, will scarcely lose all its heat in an hour's time; but a greater globe would retain its heat longer in the proportion of its diameter, because the surface (in proportion to which it is cooled by the contact of the ambient air) is in that proportion less in respect of the quantity of the included hot matter; and therefore a globe of red hot iron equal to our earth, that is, about 40000000 feet in diameter, would scarcely cool in an equal number of days, or in above 50000 years. But I suspect that the duration of heat may, on account of some latent causes, increase in a yet less proportion than that of the diameter; and I should be glad that the true proportion was investigated by experiments.

It is farther to be observed, that the comet in the month of *December*, just after it had been heated by the sun, did emit a much longer tail, and much more splendid, than in the month of *November* before, when it had not yet arrived at its perihelion; and, universally, the greatest and most fulgent tails always arise from comets immediately after their passing by the neighbourhood of the sun. Therefore the heat received by the comet conduces to the greatness of the tail: from whence, I think, I may infer, that the tail is nothing else but a very fine vapour, which the head or nucleus of the comet emits by its heat.

But we have had three several opinions about the tails of comets; for some will have it that they are nothing else but the beams of the sun's light transmitted through the comets' heads, which they suppose to be transparent; others, that they proceed from the refraction which light suffers in passing from the comet's head to the earth; and, lastly, others, that they are a sort of clouds or vapour constantly rising from the comets' heads, and tending towards the parts opposite to the sun. The first is the opinion of such as are yet unacquainted with optics; for the beams of the sun are seen in a darkened

room only in consequence of the light that is reflected from them by the little particles of dust and smoke which are always flying about in the air; and, for that reason, in air impregnated with thick smoke, those beams appear with great brightness, and move the sense vigorously; in a yet finer air they appear more faint, and are less easily discerned; but in the heavens, where there is no matter to reflect the light, they can never be seen at all. Light is not seen as it is in the beam, but as it is thence reflected to our eyes; for vision can be no otherwise produced than by rays falling upon the eyes; and, therefore, there must be some reflecting matter in those parts where the tails of the comets are seen: for otherwise, since all the celestial spaces are equally illuminated by the sun's light, no part of the heavens could appear with more splendor than another.—The second opinion is liable to many difficulties. The tails of comets are never seen variegated with those colours which commonly are inseparable from refraction; and the distinct transmission of the light of the fixed stars and planets to us is a demonstration that the æther or celestial medium is not endowed with any refractive power: for as to what is alledged, that the fixed stars have been sometimes seen by the *Egyptians* environed with a *Coma*, or *Capitulum*, because that has but rarely happened, it is rather to be ascribed to a casual refraction of clouds; and so the radiation and scintillation of the fixed stars to the refractions both of the eyes and air; for upon laying a telescope to the eye, those radiations and scintillations immediately disappear. By the tremulous agitation of the air and ascending vapours, it happens that the rays of light are alternately turned aside from the narrow space of the pupil of the eye; but no such thing can have place in the much wider aperture of the object-glass of a telescope; and hence it is that a scintillation is occasioned in the former case, which ceases in the latter; and this cessation in the latter case is a demonstration of the regular transmission of light through the heavens, without any sensible refraction. But, to obviate an objection that may be made from the appearing of no tail in such comets as shine but with a faint light, as if the secondary rays were then too

weak to affect the eyes, and for that reason it is that the tails of the fixed stars do not appear, we are to consider, that by the means of telescopes the light of the fixed stars may be augmented above an hundred fold, and yet no tails are seen; that the light of the planets is yet more copious without any tail; but that comets are seen sometimes with huge tails, when the light of their heads is but faint and dull. For so it happened in the comet of the year 1680, when in the month of *December* it was scarcely equal in light to the stars of the second magnitude, and yet emitted a notable tail, extending to the length of 40° , 50° , 60° , or 70° , and upwards; and afterwards, on the 27th and 28th of *January*, when the head appeared but as a star of the 7th magnitude, yet the tail (as we said above), with a light that was sensible enough, though faint, was stretched out to 6 or 7 degrees in length, and with a languishing light that was more difficultly seen, even to 12° , and upwards. But on the 9th and 10th of *February*, when to the naked eye the head appeared no more, through a telescope I viewed the tail of 2° in length. But farther; if the tail was owing to the refraction of the celestial matter, and did deviate from the opposition of the sun, according to the figure of the heavens, that deviation in the same places of the heavens should be always directed towards the same parts. But the comet of the year 1680, *December* 28^d. 8 $\frac{1}{2}$ ^h. P.M. at *London*, was seen in κ $8^\circ 41'$, with latitude north $28^\circ 6'$; while the sun was in φ $18^\circ 26'$. And the comet of the year 1577, *December* 29^d. was in κ $8^\circ 41'$, with latitude north $28^\circ 40'$, and the sun, as before, in about φ $18^\circ 26'$. In both cases the situation of the earth was the same, and the comet appeared in the same place of the heavens; yet in the former case the tail of the comet (as well by my observations as by the observations of others) deviated from the opposition of the sun towards the north by an angle of $4\frac{1}{2}$ degrees; whereas in the latter there was (according to the observations of *Tycho*) a deviation of 21 degrees towards the south. The refraction, therefore, of the heavens being thus disproved, it remains that the *phænomena* of the tails of comets must be derived from some reflecting matter.

And that the tails of comets do arise from their heads, and tend towards the parts opposite to the sun, is farther confirmed from the laws which the tails observe. As that, lying in the planes of the comets' orbits which pass through the sun, they constantly deviate from the opposition of the sun towards the parts which the comets' heads in their progress along these orbits have left. That to a spectator, placed in those planes, they appear in the parts directly opposite to the sun; but, as the spectator recedes from those planes, their deviation begins to appear, and daily becomes greater. That the deviation, *cæteris paribus*, appears less when the tail is more oblique to the orbit of the comet, as well as when the head of the comet approaches nearer to the sun, especially if the angle of deviation is estimated near the head of the comet. That the tails which have no deviation appear straight, but the tails which deviate are likewise bended into a certain curvature. That this curvature is greater when the deviation is greater; and is more sensible when the tail, *cæteris paribus*, is longer; for in the shorter tails the curvature is hardly to be perceived. That the angle of deviation is less near the comet's head, but greater towards the other end of the tail; and that because the convex side of the tail regards the parts from which the deviation is made, and which lie in a right line drawn out infinitely from the sun through the comet's head. And that the tails that are long and broad, and shine with a stronger light, appear more resplendent and more exactly defined on the convex than on the concave side. Upon which accounts it is plain that the *phænomena* of the tails of comets depend upon the motions of their heads, and by no means upon the places of the heavens in which their heads are seen; and that, therefore, the tails of comets do not proceed from the refraction of the heavens, but from their own heads, which furnish the matter that forms the tail. For, as in our air, the smoke of a heated body ascends either perpendicularly if the body is at rest, or obliquely if the body is moved obliquely, so in the heavens, where all bodies gravitate towards the sun, smoke and vapour must (as we have already said) ascend from the sun, and either rise

perpendicularly if the smoking body is at rest, or obliquely if the body, in all the progress of its motion, is always leaving those places from which the upper or higher parts of the vapour had risen before; and that obliquity will be least where the vapour ascends with most velocity, to wit, near the smoking body, when that is near the sun. But, because the obliquity varies, the column of vapour will be incurvated; and because the vapour in the preceding sides is something more recent, that is, has ascended something more late from the body, it will therefore be something more dense on that side, and must on that account reflect more light, as well as be better defined. I add nothing concerning the sudden uncertain agitation of the tails of comets, and their irregular figures, which authors sometimes describe, because they may arise from the mutations of our air, and the motions of our clouds, in part obscuring those tails; or, perhaps, from parts of the *Via Lactea*, which might have been confounded with and mistaken for parts of the tails of the comets as they passed by.

But that the atmospheres of comets may furnish a supply of vapour great enough to fill such immense spaces, we may easily understand from the rarity of our own air; for the air near the surface of our earth possesses a space 850 times greater than water of the same weight; and therefore a cylinder of air 850 feet high is of equal weight with a cylinder of water of the same breadth, and but one foot high. But a cylinder of air reaching to the top of the atmosphere is of equal weight with a cylinder of water about 33 feet high: and, therefore, if from the whole cylinder of air the lower part of 850 feet high is taken away, the remaining upper part will be of equal weight with a cylinder of water 32 feet high: and from thence (and by the hypothesis, confirmed by many experiments, that the compression of air is as the weight of the incumbent atmosphere, and that the force of gravity is reciprocally as the square of the distance from the centre of the earth) raising a calculus, by cor. prop. 22, book 2, I found, that, at the height of one semidiameter of the earth, reckoned from the earth's surface, the air is more rare than with us in a far greater proportion than of the whole

space within the orb of Saturn to a spherical space of one inch in diameter; and therefore if a sphere of our air of but one inch in thickness was equally rarefied with the air at the height of one semidiameter of the earth from the earth's surface, it would fill all the regions of the planets to the orb of Saturn, and far beyond it. Wherefore since the air at greater distances is immensely rarefied, and the *coma* or atmosphere of comets is ordinarily about ten times higher, reckoning from their centres, than the surface of the nucleus, and the tails rise yet higher, they must therefore be exceedingly rare: and though, on account of the much thicker atmospheres of comets, and the great gravitation of their bodies towards the sun, as well as of the particles of their air and vapours mutually one towards another, it may happen that the air in the celestial spaces and in the tails of comets is not so vastly rarefied, yet from this computation it is plain that a very small quantity of air and vapour is abundantly sufficient to produce all the appearances of the tails of comets; for that they are, indeed, of a very notable rarity appears from the shining of the stars through them. The atmosphere of the earth, illuminated by the sun's light, though but of a few miles in thickness, quite obscures and extinguishes the light not only of all the stars, but even of the moon itself; whereas the smallest stars are seen to shine through the immense thickness of the tails of comets, likewise illuminated by the sun, without the least diminution of their splendour. Nor is the brightness of the tails of most comets ordinarily greater than that of our air, an inch or two in thickness, reflecting in a darkened room the light of the sun-beams let in by a hole of the window-shutter.

And we may pretty nearly determine the time spent during the ascent of the vapour from the comet's head to the extremity of the tail, by drawing a right line from the extremity of the tail to the sun, and marking the place where that right line intersects the comet's orbit; for the vapour that is now in the extremity of the tail, if it has ascended in a right line from the sun, must have begun to rise from the head at the time when the head was in the point of intersection. It is

true, the vapour does not rise in a right line from the sun, but, retaining the motion which it had from the comet before its ascent, and compounding that motion with its motion of ascent, arises obliquely; and, therefore, the solution of the problem will be more exact, if we draw the line which intersects the orbit parallel to the length of the tail; or rather (because of the curvilinear motion of the comet) diverging a little from the line or length of the tail. And by means of this principle I found that the vapour which, *January 25*, was in the extremity of the tail, had begun to rise from the head before *December 11*, and therefore had spent in its whole ascent 45 days; but that the whole tail which appeared on *December 10* had finished its ascent in the space of the two days then elapsed from the time of the comet's being in its perihelion. The vapour, therefore, about the beginning and in the neighbourhood of the sun rose with the greatest velocity, and afterwards continued to ascend with a motion constantly retarded by its own gravity; and the higher it ascended, the more it added to the length of the tail; and while the tail continued to be seen, it was made up of almost all that vapour which had risen since the time of the comet's being in its perihelion; nor did that part of the vapour which had risen first, and which formed the extremity of the tail, cease to appear, till its too great distance, as well from the sun, from which it received its light, as from our eyes, rendered it invisible. Whence also it is that the tails of other comets which are short do not rise from their heads with a swift and continual motion, and soon after disappear, but are permanent and lasting columns of vapours and exhalations, which, ascending from the heads with a slow motion of many days, and partaking of the motion of the heads which they had from the beginning, continue to go along together with them through the heavens. From whence again we have another argument proving the celestial spaces to be free, and without resistance, since in them not only the solid bodies of the planets and comets, but also the extremely rare vapours of comets' tails, maintain their rapid motions with great freedom, and for an exceedingly long time.

Kepler ascribes the ascent of the tails of comets to the atmospheres of their heads; and their direction towards the parts opposite to the sun to the action of the rays of light carrying along with them the matter of the comets' tails; and without any great incongruity we may suppose, that, in so free spaces, so fine a matter as that of the æther may yield to the action of the rays of the sun's light, though those rays are not able sensibly to move the gross substances in our parts, which are clogged with so palpable a resistance. Another author thinks that there may be a sort of particles of matter endowed with a principle of levity, as well as others are with a power of gravity; that the matter of the tails of comets may be of the former sort, and that its ascent from the sun may be owing to its levity; but, considering that the gravity of terrestrial bodies is as the matter of the bodies, and therefore can be neither more nor less in the same quantity of matter, I am inclined to believe that this ascent may rather proceed from the rarefaction of the matter of the comets' tails. The ascent of smoke in a chimney is owing to the impulse of the air with which it is entangled. The air rarefied by heat ascends, because its specific gravity is diminished, and in its ascent carries along with it the smoke with which it is engaged; and why may not the tail of a comet rise from the sun after the same manner? For the sun's rays do not act upon the mediums which they pervade otherwise than by reflection and refraction; and those reflecting particles heated by this action, heat the matter of the æther which is involved with them. That matter is rarefied by the heat which it acquires, and because, by this rarefaction, the specific gravity with which it tended towards the sun before is diminished, it will ascend therefrom, and carry along with it the reflecting particles of which the tail of the comet is composed. But the ascent of the vapours is further promoted by their circumgyration about the sun, in consequence whereof they endeavour to recede from the sun, while the sun's atmosphere and the other matter of the heavens are either altogether quiescent, or are only moved with a slower circumgyration derived from the rotation of the sun. And these are the causes of the

ascent of the tails of the comets in the neighbourhood of the sun, where their orbits are bent into a greater curvature, and the comets themselves are plunged into the denser and therefore heavier parts of the sun's atmosphere: upon which account they do then emit tails of an huge length; for the tails which then arise, retaining their own proper motion, and in the mean time gravitating towards the sun, must be revolved in ellipses about the sun in like manner as the heads are, and by that motion must always accompany the heads, and freely adhere to them. For the gravitation of the vapours towards the sun can no more force the tails to abandon the heads, and descend to the sun, than the gravitation of the heads can oblige them to fall from the tails. They must by their common gravity either fall together towards the sun, or be retarded together in their common ascent therefrom; and, therefore (whether from the causes already described, or from any others), the tails and heads of comets may easily acquire and freely retain any position one to the other, without disturbance or impediment from that common gravitation.

The tails, therefore, that rise in the perihelion positions of the comets will go along with their heads into far remote parts, and together with the heads will either return again from thence to us, after a long course of years, or rather will be there rarefied, and by degrees quite vanish away; for afterwards, in the descent of the heads towards the sun, new short tails will be emitted from the heads with a slow motion; and those tails by degrees will be augmented immensely, especially in such comets as in their perihelion distances descend as low as the sun's atmosphere; for all vapour in those free spaces is in a perpetual state of rarefaction and dilatation; and from hence it is that the tails of all comets are broader at their upper extremity than near their heads. And it is not unlikely but that the vapour, thus perpetually rarefied and dilated, may be at last dissipated and scattered through the whole heavens, and by little and little be attracted towards the planets by its gravity, and mixed with their atmosphere; for as the seas are absolutely necessary to the constitution of our earth, that

from them, the sun, by its heat, may exhale a sufficient quantity of vapours, which, being gathered together into clouds, may drop down in rain, for watering of the earth, and for the production and nourishment of vegetables; or, being condensed with cold on the tops of mountains (as some philosophers with reason judge), may run down in springs and rivers; so for the conservation of the seas, and fluids of the planets, comets seem to be required, that, from their exhalations and vapours condensed, the wastes of the planetary fluids spent upon vegetation and putrefaction, and converted into dry earth, may be continually supplied and made up; for all vegetables entirely derive their growths from fluids, and afterwards, in great measure, are turned into dry earth by putrefaction; and a sort of slime is always found to settle at the bottom of putrefied fluids; and hence it is that the bulk of the solid earth is continually increased; and the fluids, if they are not supplied from without, must be in a continual decrease, and quite fail at last. I suspect, moreover, that it is chiefly from the comets that spirit comes, which is indeed the smallest but the most subtle and useful part of our air, and so much required to sustain the life of all things with us.

The atmospheres of comets, in their descent towards the sun, by running out into the tails, are spent and diminished, and become narrower, at least on that side which regards the sun; and in receding from the sun, when they less run out into the tails, they are again enlarged, if *Hevelius* has justly marked their appearances. But they are seen least of all just after they have been most heated by the sun, and on that account then emit the longest and most resplendent tails; and, perhaps, at the same time, the nuclei are environed with a denser and blacker smoke in the lowermost parts of their atmosphere; for smoke that is raised by a great and intense heat is commonly the denser and blacker. Thus the head of that comet which we have been describing, at equal distances both from the sun and from the earth, appeared darker after it had passed by its perihelion than it did before; for in the month of *December* it was commonly compared with the stars of the third magnitude, but in *November* with those of the

first or second ; and such as saw both appearances have described the first as of another and greater comet than the second. For, *November* 19, this comet appeared to a young man at *Cambridge*, though with a pale and dull light, yet equal to *Spica Virginis* ; and at that time it shone with greater brightness than it did afterwards. And *Montenari*, *November* 20, ft. vet., observed it larger than the stars of the first magnitude, its tail being then 2 degrees long. And Mr. *Storer* (by letters which have come into my hands) writes, that in the month of *December*, when the tail appeared of the greatest bulk and splendour, the head was but small, and far less than that which was seen in the month of *November* before sun-rising ; and, conjecturing at the cause of the appearance, he judged it to proceed from there being a greater quantity of matter in the head at first, which was afterwards gradually spent.

And, which farther makes for the same purpose, I find that the heads of other comets, which did put forth tails of the greatest bulk and splendour, have appeared but obscure and small. For in *Brasil*, *March* 5, 1668, 7^h. P. M., *St. N. P. Valentinus Estancius* saw a comet near the horizon, and towards the south west, with a head so small as scarcely to be discerned, but with a tail above measure splendid, so that the reflection thereof from the sea was easily seen by those who stood upon the shore ; and it looked like a fiery beam extended 23° in length from the west to south, almost parallel to the horizon. But this excessive splendour continued only three days, decreasing apace afterwards ; and while the splendour was decreasing, the bulk of the tail increased : whence in *Portugal* it is said to have taken up one quarter of the heavens, that is, 45 degrees, extending from west to east with a very notable splendour, though the whole tail was not seen in those parts, because the head was always hid under the horizon : and from the increase of the bulk and decrease of the splendour of the tail, it appears that the head was then in its recess from the sun, and had been very near to it in its perihelion, as the comet of 1680 was. And we read, in the *Saxon Chronicle*, of a like comet appearing in the year 1106,

the star whereof was small and obscure (as that of 1680), but the splendour of its tail was very bright, and like a huge fiery beam stretched out in a direction between the east and north, as Hevelius has it also from Simeon, the monk of Durham. This comet appeared in the beginning of February, about the evening, and towards the south west part of heaven; from whence, and from the position of the tail, we infer that the head was near the sun. Matthew Paris says, It was distant from the sun by about a cubit, from three of the clock (rather fix) till nine, putting forth a long tail. Such also was that most resplendent comet described by Aristotle, lib. 1, Meteor. 6. The head whereof could not be seen, because it had set before the sun, or at least was hid under the sun's rays; but next day it was seen as well as might be; for, having left the sun but a very little way, it set immediately after it. And the scattered light of the head, obscured by the too great splendour (of the tail), did not yet appear. But afterwards (as Aristotle says) when the splendour (of the tail) was now diminished (the head of), the comet recovered its native brightness; and the splendour (of its tail) reached now to a third part of the heavens (that is, to 60°). This appearance was in the winter season (an. 4, Olymp. 101), and, rising to Orion's girdle, it there vanished away. It is true that the comet of 1618, which came out directly from under the sun's rays with a very large tail, seemed to equal, if not to exceed, the stars of the first magnitude; but, then, abundance of other comets have appeared yet greater than this, that put forth shorter tails; some of which are said to have appeared as big as Jupiter, others as large as Venus, or even as the moon.

We have said, that comets are a sort of planets revolved in very eccentric orbits about the sun; and as, in the planets which are without tails, those are commonly less which are revolved in lesser orbits, and nearer to the sun, so in comets it is probable that those which in their perihelion approach nearer to the sun are generally of less magnitude, that they may not agitate the sun too much by their attractions. But as to the transverse diameters of their orbits, and the periodic times of their revolutions, I leave them to be determined by

comparing comets together which after long intervals of time return again in the same orbit. In the mean time, the following proposition may give some light in that enquiry.

PROPOSITION XLII. PROBLEM XXII.

To correct a comet's trajectory found as above.

OPERATION 1. Assume that position of the plane of the trajectory which was determined according to the preceding proposition; and select three places of the comet, deduced from very accurate observations, and at great distances one from the other. Then suppose A to represent the time between the first observation and the second, and B the time between the second and the third; but it will be convenient that in one of those times the comet be in its perigeon, or at least not far from it. From those apparent places find, by trigonometric operations, the three true places of the comet in that assumed plane of the trajectory; then through the places found, and about the centre of the sun as the focus, describe a conic section by arithmetical operations, according to prop. 21, book 1. Let the areas of this figure which are terminated by radii drawn from the sun to the places found be D and E; to wit, D the area between the first observation and the second, and E the area between the second and third; and let T represent the whole time in which the whole area D + E should be described with the velocity of the comet found by prop. 16, book 1.

OPER. 2. Retaining the inclination of the plane of the trajectory to the plane of the ecliptic, let the longitude of the nodes of the plane of the trajectory be increased by the addition of 20 or 30 minutes, which call P. Then from the aforesaid three observed places of the comet let the three true places be found (as before) in this new plane; as also the orbit passing through those places, and the two areas of the same described between the two observations, which call d and e; and let t be the whole time in which the whole area d + e should be described.

OPER. 3. Retaining the longitude of the nodes in the first operation, let the inclination of the plane of the trajectory to the plane of the ecliptic be increased by adding thereto 20' or

30', which call Q . Then from the aforefaid three observed apparent places of the comet let the three true places be found in this new plane, as well as the orbit paſſing through them, and the two areas of the ſame deſcribed between the obſervation, which call δ and ϵ ; and let τ be the whole time in which the whole area $\delta + \epsilon$ ſhould be deſcribed.

Then, taking C to 1 as A to B ; and G to 1 as D to E ; and g to 1 as d to e ; and γ to 1 as δ to ϵ ; let S be the true time between the firſt obſervation and the third; and, obſerving well the ſigns $+$ and $-$, let ſuch numbers m and n be found out as will make $2G - 2C; = mG - mg + nG - n\gamma$; and $2T - 2S = mT - mt + nT - n\tau$. And if, in the firſt operation, I represents the inclination of the plane of the trajectory to the plane of the ecliptic, and K the longitude of either node, then $I + nQ$ will be the true inclination of the plane of the trajectory to the plane of the ecliptic, and $K + mP$ the true longitude of the node. And, laſtly, if in the firſt, ſecond, and third operations, the quantities, R , r , and ρ , represent the parameters of the trajectory, and the quantities $\frac{1}{L}, \frac{1}{l}, \frac{1}{\lambda}$, the tranſverſe diameters of the ſame, then $R + mr - mR + n\rho - nR$ will be the true parameter, and

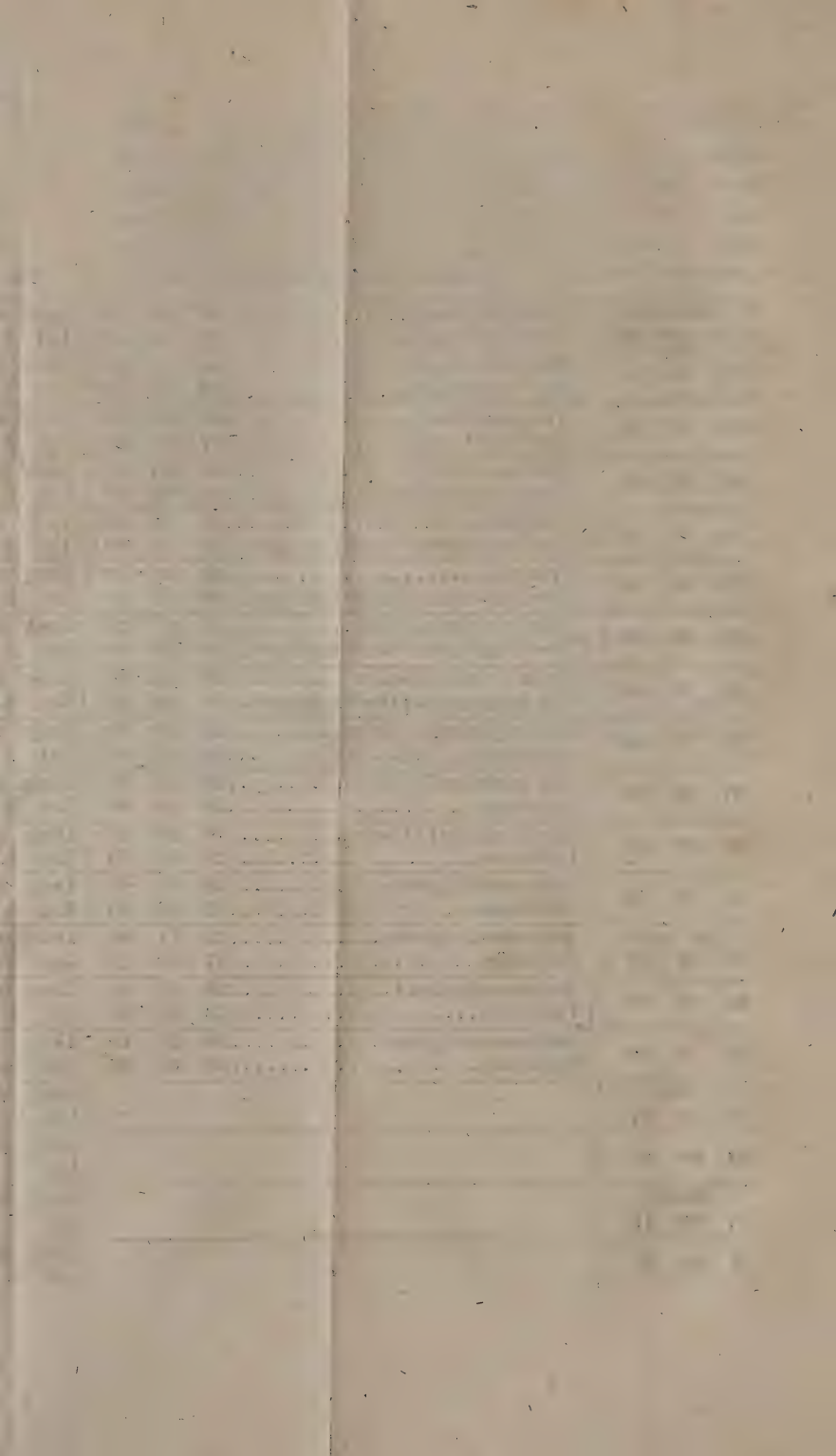
$\frac{1}{L + ml - mL + n\lambda - nL}$ will be the true tranſverſe diameter of the trajectory which the comet deſcribes; and from the tranſverſe diameter given the periodic time of the comet is alſo given. Q.E.I. But the periodic times of the revolutions of comets, and the tranſverſe diameters of their orbits, cannot be accurately enough determined but by comparing comets together which appear at different times. If, after equal intervals of time, ſeveral comets are found to have deſcribed the ſame orbit, we may thence conclude that they are all but one and the ſame comet revolved in the ſame orbit; and then from the times of their revolutions the tranſverſe diameters of their orbits will be given, and from thoſe diameters the elliptic orbits themſelves will be determined.

To this purpoſe the trajectories of many comets ought to be computed, ſuppoſing thoſe trajectories to be parabolic;

for such trajectories will always nearly agree with the *phænomena*, as appears not only from the parabolic trajectory of the comet of the year 1680, which I compared above with the observations, but likewise from that of the notable comet which appeared in the years 1664 and 1665, and was observed by *Hevelius*, who, from his own observations, calculated the longitudes and latitudes thereof, though with little accuracy. But from the same observations *Dr. Halley* did again compute its places; and from those new places determined its trajectory, finding its ascending node in π $21^{\circ} 13' 55''$; the inclination of the orbit to the plane of the ecliptic $21^{\circ} 18' 40''$; the distance of its perihelion from the node, estimated in the comet's orbit, $49^{\circ} 27' 30''$, its perihelion in Ω $8^{\circ} 40' 30''$, with heliocentric latitude south $16^{\circ} 01' 45''$; the comet to have been in its perihelion *November* 24^d. 11^h. 52' P.M. equal time at *London*, or 13^h. 8' at *Dantzic*, O.S.; and that the *latus rectum* of the parabola was 410286 such parts as the sun's mean distance from the earth is supposed to contain 100000. And how nearly the places of the comet computed in this orbit agree with the observations, will appear from the annexed table, calculated by *Dr. Halley*.

[To face Page 300.]

Apparent Time at Dantzic.	The observed Distances of the Comet from	The observed Places.	The Places com- puted in the Orb.
<i>December.</i>	The Lion's heart.....46° 24' 20"	Long. $\triangle$ 7 ^d . 01' 00"	$\triangle$ 7° 01' 29"
3 ^d . 18 ^h . 29' $\frac{1}{2}$	The Virgin's spike22 52 10	Lat. S. 21 39 00	$\triangle$ 21 38 50
4 18 01 $\frac{1}{2}$	The Lion's heart.....46 02 45	Long. $\triangle$ 6 15 00	$\triangle$ 6 16 05
	The Virgin's spike23 52 40	Lat. S. 22 24 00	22 24 00
7 17 48	The Lion's heart.....44 48 00	Long. $\triangle$ 3 06 00	$\triangle$ 3 07 33
	The Virgin's spike27 56 40	Lat. S. 25 22 00	25 21 40
17 14 43	The Lion's heart.....53 15 15	Long. Ω 2 56 00	Ω 2 56 00
	Orion's right shoulder.....45 43 30	Lat. S. 49 25 00	49 25 00
19 09 25	Procyon35 13 50	Long. Π 28 40 30	Π 28 43 00
	The bright star of the Whale's jaw..52 56 00	Lat. S. 45 48 00	45 46 00
20 09 53 $\frac{1}{2}$	Procyon40 49 00	Long. Π 13 03 00	Π 13 05 00
	The bright star of the Whale's jaw..40 04 00	Lat. S. 39 54 00	39 53 00
21 09 09 $\frac{1}{2}$	Orion's right shoulder.....26 21 25	Long. Π 2 16 00	Π 2 18 30
	The bright star of the Whale's jaw..29 28 00	Lat. S. 33 41 00	38 39 40
22 09 00	Orion's right shoulder.....29 47 00	Long. γ 24 24 00	γ 24 27 00
	The bright star of the Whale's jaw..20 29 30	Lat. S. 27 45 00	27 46 00
26 07 58	The bright star of Aries.....23 20 00	Long. γ 9 00 00	γ 9 02 28
	Aldebaran.....26 44 00	Lat. S. 12 36 00	12 34 13
27 06 45	The bright star of Aries.....20 45 00	Long. γ 7 05 40	γ 7 08 45
	Aldebaran.....28 10 00	Lat. S. 10 23 00	10 23 13
28 07 39	The bright star of Aries.....18 29 00	Long. γ 5 24 45	γ 5 27 52
	Palilicium.....29 37 00	Lat. S. 8 22 50	8 23 37
31 06 45	Andromeda's girdle.....30 48 10	Long. γ 2 07 40	γ 2 08 20
	Palilicium.....32 53 30	Lat. S. 4 13 00	4 16 25
<i>Jan. 1665.</i>	Andromeda's girdle25 11 00	Long. γ 28 24 47	γ 28 24 00
7 07 37 $\frac{1}{2}$	Palilicium.....37 12 25	Lat. N. 0 54 00	0 53 00
13 07 00	Andromeda's head.....28 07 10	Long. γ 27 06 54	γ 27 06 39
	Palilicium.....38 55 20	Lat. N. 3 06 50	3 07 40
24 7 29	Andromeda's girdle20 32 15	Long. γ 26 29 15	γ 26 28 50
	Palilicium.....40 05 00	Lat. N. 5 25 56	5 26 00
<i>Feb.</i>		Long. γ 27 04 46	γ 27 24 55
7 08 37		Lat. N. 7 03 29	7 03 15
22 08 46		Long. γ 28 29 46	γ 28 29 58
		Lat. N. 8 12 36	8 10 25
<i>March.</i>		Long. γ 29 18 15	γ 29 18 20
1 08 16		Lat. N. 8 36 26	8 36 12
7 08 37		Long. γ 0 02 48	γ 0 2 42
		Lat. N. 8 56 30	8 56 56



In *February*, the beginning of the year 1665, the first star of Aries, which I shall hereafter call γ , was in γ $28^{\circ} 30' 15''$, with $7^{\circ} 8' 58''$ north lat.; the second star of Aries was in γ $29^{\circ} 17' 18''$, with $8^{\circ} 28' 16''$ north lat.; and another star of the seventh magnitude, which I name A, was in γ $28^{\circ} 24' 45''$, with $8^{\circ} 28' 33''$ north lat. The comet, *Feb.* 7^d. 7^h. 30' at *Paris* (that is, *Feb.* 7^d. 8^h. 37' at *Dantzic*), O.S. made a triangle with those stars γ and A, which was right-angled in γ ; and the distance of the comet from the star γ was equal to the distance of the stars γ and A, that is, $1^{\circ} 19' 46''$ of a great circle; and therefore in the parallel of the latitude of the star γ it was $1^{\circ} 20' 26''$. Therefore if from the longitude of the star γ there be subducted the longitude $1^{\circ} 20' 26''$, there will remain the longitude of the comet γ $27^{\circ} 9' 49''$. M. *Auzout*, from this observation of his, placed the comet in γ $27^{\circ} 0'$, nearly; and, by the scheme in which Dr. *Hooke* delineated its motion, it was then in γ $26^{\circ} 59' 24''$. I place it in γ $27^{\circ} 4' 46''$, taking the middle between the two extremes.

From the same observation, M. *Auzout* made the latitude of the comet at that time 7° and $4'$ or $5'$ to the north; but he had done better to have made it $7^{\circ} 3' 29''$, the difference of the latitudes of the comet and the star γ being equal to the difference of the longitude of the stars γ and A.

February 22^d. 7^h. 30' at *London*, that is *February* 22^d. 8^h. 46' at *Dantzic*, the distance of the comet from the star A, according to Dr. *Hooke's* observation, as was delineated by himself in a scheme, and also by the observations of M. *Auzout*, delineated in like manner by M. *Petit*, was a fifth part of the distance between the star A and the first star of Aries, or $15' 57''$; and the distance of the comet from a right line joining the star A and the first of Aries was a fourth part of the same fifth part, that is, $4'$; and therefore the comet was in γ $28^{\circ} 29' 46''$, with $8^{\circ} 12' 36''$ north latitude.

1 *March*, 7^h. 0' at *London*, that is, *March* 1, 8^h. 16' at *Dantzic*, the comet was observed near the second star in Aries, the distance between them being to the distance between the first and second stars in Aries, that is, to $1^{\circ} 33'$, as 4 to 45

according to Dr. *Hooke*, or as 2 to 23 according to M. *Gottignies*. And, therefore, the distance of the comet from the second star in Aries was $18' 16''$ according to Dr. *Hooke*, or $8' 5''$ according to M. *Gottignies*; or, taking a mean between both, $8' 10''$. But, according to M. *Gottignies*, the comet had gone beyond the second star of Aries about a fourth or a fifth part of the space that it commonly went over in a day, to wit, about $1' 35''$ (in which he agrees very well with M. *Auzout*) ; or, according to Dr. *Hooke*, not quite so much, as perhaps only $1'$. Wherefore if to the longitude of the first star in Aries we add $1'$, and $8' 10''$ to its latitude, we shall have the longitude of the comet γ $29^{\circ} 18'$, with $8^{\circ} 36' 26''$ north latitude.

7 March, 7^h. 30' at *Paris* (that is, March 7, 8^h. 37' at *Dantzic*), from the observations of M. *Auzout*, the distance of the comet from the second star in Aries was equal to the distance of that star from the star A, that is, $52' 29''$; and the difference of the longitude of the comet and the second star in Aries was $45'$ or $46'$, or, taking a mean quantity, $45' 30''$; and therefore the comet was in γ $0^{\circ} 2' 48''$. From the scheme of the observations of M. *Auzout*, constructed by M. *Petit*, *Hevelius* collected the latitude of the comet $8^{\circ} 54'$. But the engraver did not rightly trace the curvature of the comet's way toward the end of the motion; and *Hevelius*, in the scheme of M. *Auzout*'s observations which he constructed himself, corrected this irregular curvature, and so made the latitude of the comet $8^{\circ} 55' 30''$. And, by farther correcting this irregularity, the latitude may become $8^{\circ} 56'$, or $8^{\circ} 57'$.

This comet was also seen March 9, and at that time its place must have been in γ $0^{\circ} 18'$, with $9^{\circ} 3\frac{1}{2}'$ north latitude, nearly.

This comet appeared three months together, in which space of time it travelled over almost six signs, and in one of the days thereof described almost 20 deg. Its course did very much deviate from a great circle, bending towards the north, and its motion towards the end from retrograde became direct; and, notwithstanding its course was so uncommon, yet by the table it appears that the theory, from beginning to end, agrees with the observations no less accurately than the

theories of the planets usually do with the observations of them; but we are to subduct about $2'$ when the comet was swiftest, which we may effect by taking off $12''$ from the angle between the ascending node and the perihelion, or by making that angle $49^{\circ} 27' 18''$. The annual parallax of both these comets (this and the preceding) was very conspicuous, and by its quantity demonstrates the annual motion of the earth in the *orbis magnus*.

This theory is likewise confirmed by the motion of that comet, which in the year 1683 appeared retrograde, in an orbit whose plane contained almost a right angle with the plane of the ecliptic, and whose ascending node (by the computation of Dr. *Halley*) was in $\text{m} 23^{\circ} 23'$; the inclination of its orbit to the ecliptic $83^{\circ} 11'$; its perihelion in $\text{II} 25^{\circ} 29' 30''$; its perihelion distance from the sun 56020 of such parts as the radius of the *orbis magnus* contains 100000; and the time of its perihelion *July* 2^{d} . 3^{h} . $50'$. And the places thereof, computed by Dr. *Halley* in this orbit, are compared with the places of the same observed by Mr. *Flamsted*, in the following table.

1685 Eq. time.	Sun's place.		Comet's Long comput.		Lat. Nor comput.		Comet's Long. observ.		Lat. Nor. observed.		Diff. Long.		Diff. Lat.		
d.	h.	°	'	°	'	°	'	°	'	°	'	°	'		
July	13	12	55	♈	13	05	42	29	28	13	+1	00	+0	07	
	15	11	15	2	53	12	11	37	48	29	34	+1	55	+0	50
	17	10	20	4	45	45	10	07	06	29	33	+1	34	+0	30
	23	13	40	10	38	21	5	10	27	28	51	+1	03	-1	14
	25	14	5	12	35	28	3	27	53	24	24	-0	53	-1	07
Aug.	31	9	42	♏	27	55	03	26	22	52	-0	39	-0	27	
	31	14	55	18	21	53	27	41	07	26	16	+0	01	-2	07
	2	14	56	20	17	16	25	29	32	25	16	-0	46	+1	09
	4	10	49	22	02	50	23	18	20	24	10	-1	25	+1	30
	6	10	9	23	56	45	20	42	23	22	47	-1	51	+2	00
	9	10	26	26	50	52	16	07	57	20	06	-2	02	-0	27
	15	14	1	2	47	13	3	30	48	11	37	-4	30	-5	32
	16	15	10	3	48	02	0	43	07	9	34	-1	12	-0	03
	18	15	44	5	45	33	8	24	52	53	5	-3	48	-2	04
	22	14	44	9	35	49	11	07	14	5	16	-0	02	-0	03
23	15	52	10	36	48	7	02	18	8	17	-1	01	-0	28	
26	16	2	13	31	10	9	24	45	31	16	-1	31	+0	20	

This theory is yet farther confirmed by the motion of that retrograde comet which appeared in the year 1682. The ascending node of this (by Dr. *Halley's* computation) was in γ $21^{\circ} 16' 30''$; the inclination of its orbit to the plane of the ecliptic $17^{\circ} 56' 00''$; its perihelion in $\approx$ $2^{\circ} 52' 50''$; its perihelion distance from the sun 58328 parts, of which the radius of the *orbis magnus* contains 100000; the equal time of the comet's being in its perihelion *Sept.* 4^d. 7^h. 39'. And its places, collected from Mr. *Flamsted's* observations, are compared with its places computed from our theory in the following table.

1862 Appar. time.	Sun's place.	Comet's		Lat. Nor. comp.	Com. Long. observed.	Lat. Nor. observed.	Diff.	
		Long. comp.	Lat. comp.				Long.	Lat.
Aug. 19 16 38 20 15 38 21 08 21 22 08 08 29 08 20 30 07 45	♊ 7 00 07	♊ 18 14 28	♊ 25 50 07	0 18 14 40	♊ 25 49 55	0 12	-0 12	+0 12
	♊ 7 55 52	♊ 24 46 23	♊ 26 14 42	0 24 46 22	♊ 26 12 52	0 01	+0 01	+1 50
	♊ 8 36 14	♊ 29 37 15	♊ 26 20 03	0 29 38 02	♊ 26 17 37	0 47	+0 47	+2 26
	♊ 9 33 55	♊ 6 29 53	♊ 26 08 42	♊ 6 30 03	♊ 26 07 12	0 10	+0 10	+1 30
	♊ 16 22 40	♊ 12 37 54	♊ 18 37 47	♊ 12 37 49	♊ 18 34 05	0 05	+0 05	+3 42
	♊ 17 19 41	♊ 15 36 01	♊ 17 26 43	♊ 15 35 18	♊ 17 27 17	0 43	+0 43	-0 34
	♊ 19 16 09	♊ 20 30 53	♊ 15 13 00	♊ 20 27 04	♊ 15 09 49	3 49	+3 49	+3 11
Sept. 4 07 22 5 07 32 8 07 16 9 07 26	♊ 22 11 28	♊ 25 42 00	♊ 12 23 48	♊ 25 40 58	♊ 12 22 00	1 02	+1 02	+1 48
	♊ 23 10 29	♊ 27 00 46	♊ 11 33 08	♊ 26 59 24	♊ 11 33 51	1 22	+1 22	-0 43
	♊ 26 05 58	♊ 29 58 44	♊ 9 26 46	♊ 29 58 45	♊ 9 26 43	0 01	-0 01	+0 03
	♊ 27 05 09	♊ 0 44 10	♊ 8 49 10	♊ 0 44 04	♊ 8 48 25	0 06	+0 06	+0 45

This theory is also confirmed by the retrograde motion of the comet that appeared in the year 1723. The ascending node of this comet (according to the computation of Mr. *Bradley*, Savilian Professer of Astronomy at *Oxford*) was in ♊ 14° 16'. The inclination of the orbit to the plane of the ecliptic 49° 59'. Its perihelion was in ♊ 12° 15' 20". Its perihelion distance from the sun 998651 parts, of which the radius of the *orbis magnus* contains 1000000, and the equal time of its perihelion September 16^d. 16^h. 10'. The places of this comet computed in this orbit by Mr. *Bradley*, and compared with the places observed by himself, his uncle Mr.

Pound, and Dr. *Halley*, may be seen in the following table :—

1723. Eq. time.	Comet's Long. obs.	Lat. Nor. obs.	Comet's Long. com.	Lat. Nor. comp.	Diff. Long.	Diff. Lat.
d. h. ' "	o ' "	o ' "	o ' "	o ' "	" "	" "
Oct. 9 8 5	7 22 15	5 2 0	7 21 26	5 2 47	+49	-47
10 6 21	6 41 12	7 44 13	6 41 42	7 43 18	-50	+55
12 7 22	5 39 58	11 55 0	5 40 19	11 54 55	-21	+5
14 8 57	4 59 49	14 43 50	5 0 37	14 44 1	-48	-11
15 6 35	4 47 41	15 40 51	4 47 45	15 40 55	-4	-4
21 6 22	4 2 32	19 41 49	4 2 21	19 42 3	+11	-14
22 6 24	3 59 20	8 12	3 59 10	20 8 17	-8	-5
24 8 2	3 55 29	20 55 18	3 55 11	20 55 9	+18	+9
29 8 56	3 56 17	22 20 27	3 56 42	22 20 10	-25	+17
30 6 20	3 58 9	22 32 28	3 58 17	22 32 12	-8	+16
Nov. 5 5 53	4 16 30	23 38 33	4 16 23	23 38 7	+7	+26
8 7 6	4 29 36	24 4 30	4 29 54	24 4 40	-18	-10
14 6 20	5 2 16	24 48 46	5 2 51	24 48 16	-35	+30
20 7 45	5 42 20	25 24 45	5 43 13	25 25 17	-53	-32
Dec. 7 6 45	8 4 13	26 54 18	8 3 55	26 53 42	+18	+36

From these examples it is abundantly evident that the motions of comets are no less accurately represented by our theory than the motions of the planets commonly are by the theories of them; and, therefore, by means of this theory, we may enumerate the orbits of comets, and so discover the periodic time of a comet's revolution in any orbit; whence, at last, we shall have the transverse diameters of their elliptic orbits and their aphelion distances.

That retrograde comet which appeared in the year 1607 described an orbit whose ascending node (according to Dr. *Halley*'s computation) was in $8^{\circ} 20' 21''$; and the inclination of the plane of the orbit to the plane of the ecliptic $17^{\circ} 2'$; whose perihelion was in $2^{\circ} 16'$; and its perihelion distance from the sun 58680 of such parts as the radius of the *orbis magnus* contains 100000; and the comet was in its perihelion October 16^d. 3^h. 50'; which orbit agrees very nearly with the orbit of the comet which was seen in 1682. If these were not two different comets, but one and the same, that comet will finish one revolution in the space of 75 years; and the greater axis of its orbit will be to the greater axis of the *orbis*

magnus as $\sqrt[3]{75 \times 75}$ to 1, or as 1778 to 100, nearly. And the aphelion distance of this comet from the sun will be to the mean distance of the earth from the sun as about 35 to 1; from which data it will be no hard matter to determine the elliptic orbit of this comet. But these things are to be supposed on condition, that, after the space of 75 years, the same comet shall return again in the same orbit. The other comets seem to ascend to greater heights, and to require a longer time to perform their revolutions.

But, because of the great number of comets, of the great distance of their aphelions from the sun, and of the slowness of their motions in the aphelions, they will, by their mutual gravitations, disturb each other; so that their eccentricities and the times of their revolutions will be sometimes a little increased, and sometimes diminished. Therefore we are not to expect that the same comet will return exactly in the same orbit, and in the same periodic times: it will be sufficient if we find the changes no greater than may arise from the causes just spoken of.

And hence a reason may be assigned why comets are not comprehended within the limits of a zodiac, as the planets are; but, being confined to no bounds, are with various motions dispersed all over the heavens; namely, to this purpose, that in their aphelions, where their motions are exceedingly slow, receding to greater distances one from another, they may suffer less disturbance from their mutual gravitations: and hence it is that the comets which descend the lowest, and therefore move the slowest in their aphelions, ought also to ascend the highest.

The comet which appeared in the year 1680 was in its perihelion less distant from the sun than by a sixth part of the sun's diameter; and because of its extreme velocity in that proximity to the sun, and some density of the sun's atmosphere, it must have suffered some resistance and retardation; and therefore, being attracted something nearer to the sun in every revolution, will at last fall down upon the body of the sun. Nay, in its aphelion, where it moves the slowest, it may sometimes happen to be yet farther retarded by the attractions of

other comets, and in consequence of this retardation descend to the sun. So fixed stars, that have been gradually wasted by the light and vapours emitted from them for a long time, may be recruited by comets that fall upon them; and from this fresh supply of new fuel those old stars, acquiring new splendour, may pass for new stars. Of this kind are such fixed stars as appear on a sudden, and shine with a wonderful brightness at first, and afterwards vanish by little and little. Such was that star which appeared in *Cassiopeia's* chair; which *Cornelius Gemma* did not see upon the 8th of *November*, 1572, though he was observing that part of the heavens upon that very night, and the sky was perfectly serene; but the next night (*November* 9) he saw it shining much brighter than any of the fixed stars, and scarcely inferior to *Venus* in splendour. *Tycho Brahe* saw it upon the 11th of the same month, when it shone with the greatest lustre; and from that time he observed it to decay by little and little; and in 16 months' time it entirely disappeared. In the month of *November*, when it first appeared, its light was equal to that of *Venus*. In the month of *December* its light was a little diminished, and was now become equal to that of *Jupiter*. In *January* 1573 it was less than *Jupiter*, and greater than *Sirius*; and about the end of *February* and the beginning of *March* became equal to that star. In the months of *April* and *May* it was equal to a star of the second magnitude; in *June*, *July*, and *August*, to a star of the third magnitude; in *September*, *October*, and *November*, to those of the fourth magnitude; in *December* and *January* 1574, to those of the fifth; in *February* to those of the sixth magnitude; and in *March* it entirely vanished. Its colour at the beginning was clear, bright, and inclining to white; afterwards it turned a little yellow; and in *March* 1573 it became ruddy, like *Mars* or *Aldebaran*; in *May* it turned to a kind of dusky whiteness, like that we observe in *Saturn*; and that colour it retained ever after, but growing always more and more obscure. Such also was the star in the right foot of *Serpentarius*, which *Kepler's* scholars first observed *September* 30, O.S. 1604, with a light exceeding that of *Jupiter*;

though the night before it was not to be seen ; and from that time it decreased by little and little, and in 15 or 16 months entirely disappeared. Such a new star appearing with an unusual splendour is said to have moved *Hipparchus* to observe, and make a catalogue of, the fixed stars. As to those fixed stars that appear and disappear by turns, and increase slowly and by degrees, and scarcely ever exceed the stars of the third magnitude, they seem to be of another kind, which revolve about their axes, and, having a light and a dark side, shew those two different sides by turns. The vapours which arise from the sun, the fixed stars, and the tails of the comets, may meet at last with, and fall into, the atmospheres of the planets by their gravity, and there be condensed and turned into water and humid spirits ; and from thence, by a slow heat, pass gradually into the form of salts, and sulphurs, and tinctures, and mud, and clay, and sand, and stones, and coral, and other terrestrial substances.

GENERAL SCHOLIUM.

The hypothesis of vortices is pressed with many difficulties. That every planet by a radius drawn to the sun may describe areas proportional to the times of description, the periodic times of the several parts of the vortices should observe the duplicate proportion of their distances from the sun ; but that the periodic times of the planets may obtain the sesquiplicate proportion of their distances from the sun, the periodic times of the parts of the vortex ought to be in the sesquiplicate proportion of their distances. That the smaller vortices may maintain their lesser revolutions about *Saturn*, *Jupiter*, and other planets, and swim quietly and undisturbed in the greater vortex of the sun, the periodic times of the parts of the sun's vortex should be equal ; but the rotation of the sun and planets about their axes, which ought to correspond with the motions of their vortices, recede far from all these proportions. The motions of the comets are exceedingly regular, are governed by the same laws with the motions of the planets, and can by no means be accounted for by the hypothesis of vortices ; for comets are carried with very eccentric motions

through all parts of the heavens indifferently, with a freedom that is incompatible with the notion of a vortex.

Bodies projected in our air suffer no resistance but from the air. Withdraw the air, as is done in Mr. *Boyle's* vacuum, and the resistance ceases; for in this void a bit of fine down and a piece of solid gold descend with equal velocity. And the parity of reason must take place in the celestial spaces above the earth's atmosphere; in which spaces, where there is no air to resist their motions, all bodies will move with the greatest freedom; and the planets and comets will constantly pursue their revolutions in orbits given in kind and position, according to the laws above explained; but though these bodies may, indeed, persevere in their orbits by the mere laws of gravity, yet they could by no means have at first derived the regular position of the orbits themselves from those laws.

The six primary planets are revolved about the sun in circles concentric with the sun, and with motions directed towards the same parts, and almost in the same plane. Ten moons are revolved about the earth, Jupiter and Saturn, in circles concentric with them, with the same direction of motion, and nearly in the planes of the orbits of those planets: but it is not to be conceived that mere mechanical causes could give birth to so many regular motions, since the comets range over all parts of the heavens in very eccentric orbits; for by that kind of motion they pass easily through the orbs of the planets, and with great rapidity; and in their aphelions, where they move the slowest, and are detained the longest, they recede to the greatest distances from each other, and thence suffer the least disturbance from their mutual attractions. This most beautiful system of the sun, planets, and comets, could only proceed from the counsel and dominion of an intelligent and powerful BEING. And if the fixed stars are the centres of other like systems, these, being formed by the like wise counsel, must be all subject to the dominion of ONE; especially since the light of the fixed stars is of the same nature with the light of the sun, and from every system light passes into all the other systems: and lest the systems of the fixed stars should, by their gravity,

fall on each other mutually, he hath placed those systems at immense distances one from another.

This BEING governs all things, not as the soul of the world, but as Lord over all; and on account of his dominion he is wont to be called *Lord God* παντοκράτωρ, or *Universal Ruler*; for *God* is a relative word, and has a respect to servants; and *Deity* is the dominion of GOD not over his own body, as those imagine who fancy GOD to be the soul of the world, but over servants. The SUPREME GOD is a Being eternal, infinite, absolutely perfect; but a being, however perfect, without dominion, cannot be said to be Lord God; for we say, my God, your God, the God of *Israel*, the God of gods, and Lord of lords; but we do not say, my Eternal, your Eternal, the Eternal of *Israel*, the Eternal of gods; we do not say, my Infinite, or my Perfect: these are titles which have no respect to servants. The word *God** usually signifies *Lord*; but every lord is not a God. It is the dominion of a spiritual being which constitutes a God: a true, supreme, or imaginary dominion makes a true, supreme, or imaginary God. And from his true dominion it follows that the true GOD is a living, intelligent, and powerful Being; and, from his other perfections, that he is supreme, or most perfect. He is eternal and infinite, omnipotent and omniscient; that is, his duration reaches from eternity to eternity; his presence from infinity to infinity; he governs all things, and knows all things that are or can be done. He is not eternity or infinity, but eternal and infinite; he is not duration or space, but he endures and is present. He endures for ever, and is every where present; and by existing always and every where, he constitutes duration and space. Since every particle of space is *always*, and every indivisible moment of duration is *every where*, certainly the Maker and Lord of all things cannot be

* Dr. Pocock derives the Latin word *Deus* from the *Arabic* *du* (in the oblique case *di*), which signifies *Lord*. And in this sense princes are called *gods*, *Psal.* lxxxii, ver. 6; and *John* x, ver. 35. And *Moses* is called a *god* to his brother *Aaron*, and a *god* to *Pharaoh* (*Exod.* iv, ver. 16; and vii, ver. 8). And in the same sense the souls of dead princes were formerly, by the Heathens, called *gods*, but falsely, because of their want of dominion.

never and *no where*. Every soul that has perception is, though in different times and in different organs of sense and motion, still the same indivisible person. There are given successive parts in duration, co-existent parts in space, but neither the one nor the other in the person of a man, or his thinking principle; and much less can they be found in the thinking substance of GOD. Every man, so far as he is a thing that has perception, is one and the same man during his whole life, in all and each of his organs of sense. GOD is the same GOD, always and every where. He is omnipresent not *virtually* only, but also *substantially*; for virtue cannot subsist without substance. In him* are all things contained and moved; yet neither affects the other: GOD suffers nothing from the motion of bodies; bodies find no resistance from the omnipresence of GOD. It is allowed by all that the SUPREME GOD exists necessarily; and by the same necessity he exists *always* and *every where*. Whence also he is all similar, all eye, all ear, all brain, all arm, all power to perceive, to understand, and to act; but in a manner not at all human, in a manner not at all corporeal, in a manner utterly unknown to us. As a blind man has no idea of colours, so have we no idea of the manner by which the all-wise GOD perceives and understands all things. He is utterly void of all body and bodily figure, and can therefore neither be seen, nor heard, nor touched; nor ought he to be worshipped under the representation of any corporeal thing. We have ideas of his attributes, but what the real substance of any thing is we know not. In bodies, we see only their figures and colours, we hear only the sounds, we touch only their outward surfaces, we smell only the odours, and taste the flavours; but their in-

* This was the opinion of the Antients. So *Pythagoras*, in *Cicer. de Nat. Deor.* lib. i. *Thales Anaxagoras*, *Virgil*, *Georg.* lib. iv, ver. 220; and *Æneid*, lib. vi, ver. 721. *Philo Allegor.* at the beginning of lib. i. *Aratus*, in his *Phænomena* at the beginning. So also the sacred writers; as *St. Paul*, *Acts* xvii, ver. 27, 28. *St. John's Gosp.* chap. xiv, ver. 2. *Moses*, in *Deut.* iv, ver. 39; and x, ver. 14. *David*, *Psal.* cxxxix, ver. 7, 8, 9. *Solomon*, 1 *Kings*, viii, ver. 27. *Job* xxii, ver. 12, 13, 14. *Jeremiah* xxiii, ver. 23, 24. The Idolaters supposed the sun, moon, and stars, the souls of men, and other parts of the world, to be parts of the SUPREME GOD, and therefore to be worshipped; but erroneously.

ward substances are not to be known either by our senses, or by any reflex act of our minds : much less, then, have we any idea of the substance of GOD. We know him only by his most wise and excellent contrivances of things, and final causes ; we admire him for his perfections ; but we reverence and adore him on account of his dominion : for we adore him as his servants ; and a god without dominion, providence, and final causes, is nothing else but Fate and Nature. Blind metaphysical necessity, which is certainly the same always and every where, could produce no variety of things. All that diversity of natural things which we find suited to different times and places could arise from nothing but the ideas and will of a Being necessarily existing. But, by way of allegory, GOD is said to see, to speak, to laugh, to love, to hate, to desire, to give, to receive, to rejoice, to be angry, to fight, to frame, to work, to build ; for all our notions of GOD are taken from the ways of mankind by a certain similitude, which, though not perfect, has some likeness, however. And thus much concerning GOD ; to discourse of whom from the appearances of things does certainly belong to Natural Philosophy.

Hitherto we have explained the phænomena of the heavens and of our sea by the power of gravity, but have not yet assigned the cause of this power. This is certain, that it must proceed from a cause that penetrates to the very centres of the sun and planets, without suffering the least diminution of its force ; that operates not according to the quantity of the surfaces of the particles upon which it acts (as mechanical causes use to do), but according to the quantity of the solid matter which they contain, and propagates its virtue on all sides to immense distances, decreasing always in the duplicate proportion of the distances. Gravitation towards the sun is made up out of the gravitations towards the several particles of which the body of the sun is composed ; and in receding from the sun decreases accurately in the duplicate proportion of the distances as far as the orb of Saturn, as evidently appears from the quiescence of the aphelions of the planets ; nay, and even to the remotest aphelions of the co-

mets, if those aphelions are also quiescent. But hitherto I have not been able to discover the cause of those properties of gravity from phænomena, and I frame no hypotheses; for whatever is not deduced from the phænomena is to be called an hypothesis; and hypotheses, whether metaphysical or physical, whether of occult qualities or mechanical, have no place in experimental philosophy. In this philosophy particular propositions are inferred from the phænomena, and afterwards rendered general by induction. Thus it was that the impenetrability, the mobility, and the impulsive force of bodies, and the laws of motion and of gravitation, were discovered. And to us it is enough that gravity does really exist, and act according to the laws which we have explained, and abundantly serves to account for all the motions of the celestial bodies, and of our sea.

And now we might add something concerning a certain most subtle Spirit which pervades and lies hid in all gross bodies; by the force and action of which Spirit the particles of bodies mutually attract one another at near distances, and cohere, if contiguous; and electric bodies operate to greater distances, as well repelling as attracting the neighbouring corpuscles; and light is emitted, reflected, refracted, inflected, and heats bodies; and all sensation is excited, and the members of animal bodies move at the command of the will, namely, by the vibrations of this Spirit, mutually propagated along the solid filaments of the nerves, from the outward organs of sense to the brain, and from the brain into the muscles. But these are things that cannot be explained in few words, nor are we furnished with that sufficiency of experiments which is required to an accurate determination and demonstration of the laws by which this electric and elastic Spirit operates.

End of the Mathematical Principles.

APPENDIX.

Among the explications (given by a friend) of some propositions in this work not demonstrated by the author, the editor finding the following, has thought it proper to annex them. Thus,

To corollary 2, proposition 91, book 1, page 199.

TO find the force whereby a sphere (AdBg) on the diameter AB attracts the body P. (Pl. 19, Fig. 1.)

Let $SA = SB = r$, $PS = d$, $PE = x$, $PB = a = d + r$, $PA = \alpha = d - r$: therefore $a\alpha = dd - rr$; also $a + \alpha = 2d$, $a - \alpha = 2r$; therefore $aa - \alpha\alpha = 4dr$; and $SE = d - x$, $AE = x - \alpha$, $BE = a - x$.

Now the force whereby the circle, whose radius is Ed, attracts the body P, is as $1 - \frac{PE}{Pd}$ (by cor. 1, prop. 90.)

And $\overline{Ed}^2 = (AE \times EB = \overline{SA}^2 - \overline{SE}^2 = rr - dd + 2dx - xx) = a\alpha + 2dx - xx$. Also $\overline{Pd}^2 = (\overline{Ed}^2 + \overline{EP}^2 = 2dx - a\alpha - xx + xx =) 2dx - a\alpha$: therefore $\frac{PE}{Pd} =$

$\frac{x}{\sqrt{-a\alpha + 2dx}}$. Therefore $1 - \frac{x}{\sqrt{-a\alpha + 2dx}}$, or $\dot{x} -$

$\frac{xx}{\sqrt{-a\alpha + 2dx}}$ is the fluxion of the attractive force of the sphere on the body P, or the ordinate of a curve whose area represents that force.

But the fluent of $\dot{x}$ is x ; and the fluent of $\frac{xx}{\sqrt{-a\alpha + 2dx}}$ is $\frac{a\alpha + dx}{3dd} \sqrt{-a\alpha + 2dx}$ (by Table 1, Form 4, Case 2, Quadrature of Curves).

Therefore $x - \frac{a\alpha + dx}{3dd} \sqrt{-a\alpha + 2dx}$ is the general expression of the area of the curve.

Now let $x = a$, then area $= (a - \frac{a\alpha + da}{3dd} \sqrt{-a\alpha + 2da})$
 $\Rightarrow \frac{d^3 + r^3}{3dd} = A.$

Also let $x = \alpha$, then area $= (\alpha - \frac{a\alpha + d\alpha}{3dd} \sqrt{-a\alpha + 2d\alpha})$
 $\Rightarrow \frac{d^3 - r^3}{3dd} = B.$

And the force whereby the sphere attracts the body P is as
 $(A - B, \text{ or as } \frac{2r^3}{3d^2} =) \frac{2SA^3}{3PS^2}.$

2. The force whereby the spheroid ADBG attracts the body P, may, in the same manner, be found thus. Let $SC = c$.

The force of a circle whose radius is ED to attract P is as
 $1 - \frac{PE}{PD}$ (by cor. I, prop. 90.) Now $\overline{ED}^2 = \frac{\overline{SC}^2}{\overline{SA}^2} \times \overline{AEB} =$
 $\frac{cc}{rr} \times -a\alpha + 2dx - xx$ (by the conics); and $\overline{PD}^2 = (\overline{ER}^2$
 $= \overline{ED}^2 + \overline{EP}^2 = \frac{-a\alpha cc + 2d\alpha cc - ccxx}{rr} + xx =)$
 $\frac{-a\alpha cc + 2d\alpha cc + rr - cc \times x^2}{rr}.$ Therefore $(1 -$

$\frac{PE}{PD} = 1 - \frac{x}{\sqrt{-\frac{a\alpha cc}{rr} + \frac{2d\alpha cc}{rr} x + \frac{rr - cc}{rr} xx}}, \text{ or } \frac{x}{\sqrt{-\frac{a\alpha cc}{rr} + \frac{2d\alpha cc}{rr} x + \frac{rr - cc}{rr} xx}}$ is the fluxion of

the attractive force of the spheroid on the body P, or the ordinate of a curve whose area is the measure of that force.

Now the fluent of x is x ; and (by *Case 2, Form 8, Table 2,*

Quad. Cur.) the fluent of $\sqrt{-\frac{aacc}{rr} + \frac{2dcc}{rr}x + \frac{rr-cc}{ra}xx}$
 is $\left(-\frac{8dcc}{rr}s + \frac{4dcc}{rr}xv - \frac{4aacc}{rr}v - \frac{2drrs + drrxv - aarrv}{aa \times cc - rr - ddcc} \right)$
 $= \frac{-2ds + dxv - aav}{-cc - dd + rr} = \frac{2ds - dxv + aav}{cc + dd - rr}$. Therefore
 $x + \frac{dxv - aav - 2ds}{cc + dd - rr}$ is the general expression for the area
 of the curve.

But $v = PD = ER = \sqrt{-\frac{aacc}{rr} + \frac{2dcc}{rr}x + \frac{cc-rr}{rr}xx}$

is an ordinate to a conic section whose abscissa is x ; and s, σ ,
 the areas NMB, NKA, adjacent to the ordinates BM, AK.
 Put $D = s - \sigma$.

Let $x = a$, or $PE = PB = BM$; then $v = a$, or $PD =$
 $PB = BM$, and the area $= a + \frac{daa - aaa - 2ds}{cc + dd - rr} = A$. And
 let $x = \alpha$ or $PE = PA = AK$; then $v = \alpha$, or $PD = PA$
 $= AK$, and the area $= \alpha + \frac{d\alpha\alpha - a\alpha\alpha - 2d\sigma}{cc + dd - rr} = B$.

And the attractive force of the spheroid on P is as $(A - B)$
 $= a - \alpha + \frac{d \times aa - a\alpha - a\alpha \times a - \alpha - 2d \times s - \sigma}{cc + dd - rr}$
 $= 2r + \frac{2ddr + 2r^3 - 2dD}{cc + dd - rr} = \frac{2rcc + 2d \times 2dr - D}{cc + dd - rr}$.

But $2d = (a + \alpha) = BM + AK$, therefore $2dr =$
 trapezium ABMK; and $D = (s - \sigma) =$ area AKRMB;
 therefore $D - 2dr =$ mixtilinear area KRMLK $= C$;
 consequently $2dr - D = -C$; therefore $2d \times 2dr - D$
 $= -2dC$; therefore the attractive force of the spheroid
 on P is as $\frac{2rcc - 2dC}{cc + dd - rr} = \frac{2AS \times SC^2 - 2PS \times KRMK}{SC^2 + PS^2 - AS^2}$.

Consequently the attractive force of the spheroid upon the

body P will be to the attractive force of a sphere whose diameter is AB upon the same body P as $\frac{rcc - dC}{cc + dd - rr}$ to $\frac{r^3}{3dd}$,

or as $\frac{AS \times \overline{SC}^2 - PS \times KRMK}{\overline{SC}^2 + \overline{PS}^2 - \overline{AS}^2}$ to $\frac{\overline{AS}^3}{3\overline{PS}^2}$.

To Scholium, proposition 34, book 2, page 94.

For let it be proposed to find the vertex of the cone, a frustum of which has the described property.

Let CFGB be the frustum, and S the vertex required. (Pl. 19, Fig. 2.)

Now conceive the medium to consist of particles which strike the surface of a body (moving in it) in a direction opposite to that of the motion; then the resistance will be the force which is made up of the efficacy of the forces of all the strokes.

In any line Pp, parallel to the axis of the cone, and meeting its surface in p, take pm of a given length, for the space described by each point of the cone in a given time: draw mq perpendicular to the side (CF) of the cone, and qn perpendicular to pm.

Therefore the line pm will represent the velocity, or force, with which a particle of the medium strikes the surface of the cone obliquely in p.

But the force mp is equivalent to two forces, the one (mq) perpendicular, the other (pq) parallel to the side of the cone; which last is therefore of no effect.

And the perpendicular force mq is equivalent to two forces, the one (mn) parallel to the axis of the cone, the other (qn) perpendicular to it; which also is destroyed by the contrary action of another particle on the opposite side of the cone.

There remains only the force mn, which has any effect in resisting or moving the cone in the direction of its axis.

Therefore the whole force of a single particle, or the effect of the perpendicular stroke of a particle, upon the base of a circumscribing cylinder, is to the effect of the oblique stroke

upon the surface of the cone (in p) as mp to mn, or as $\overline{mp^2}$ to $(mp \times mn =) \overline{mq^2}$, or as $\overline{CF^2}$ to $\overline{CH^2}$.

Now the number of particles striking in a parallel direction on any surface is as the area of a plane figure perpendicular to that direction, and that would just receive those strokes.

Therefore the number of particles striking against the frustum, that is, against the surfaces described by the rotation of FD and CF, each particle with the forces mp and mn respectively, is as the circle described by (FD or) OH, and the annulus described by CH, that is, as $\overline{OH^2}$ to $\overline{CO^2} - \overline{OH^2}$.

But the whole force of the medium in resisting is the sum of the forces of the several particles.

Therefore the resistance of the medium, or the whole efficacy of the force of all the strokes against the end FG of the frustum, is to the resistance against the convex surface thereof as $(mp \times \overline{OH^2} \text{ to } mn \times \overline{CO^2} - \overline{OH^2}, \text{ or as } \overline{CF^2} \times \overline{OH^2} \text{ to } \overline{CH^2}$

$$\times \overline{CO^2} - \overline{OH^2}, \text{ or as) } \overline{OH^2} \text{ to } \frac{\overline{CH^2} \times \overline{CO^2} - \overline{OH^2}}{\overline{CF^2}}.$$

Therefore the whole resistance of the medium against the frustum may be represented by $(\overline{OH^2} \times \frac{\overline{CH^2} \times \overline{CO^2} - \overline{OH^2}}{\overline{CF^2}})$

$$= \frac{\overline{CF^2} \times \overline{OH^2} - \overline{CH^2} \times \overline{OH^2} + \overline{CH^2} \times \overline{OC^2}}{\overline{CF^2}} =)$$

$$\frac{\overline{HF^2} \times \overline{OH^2} + \overline{CH^2} \times \overline{OC^2}}{\overline{CF^2}}, \text{ which call } z; \text{ that is, putting}$$

$$OC = r, OD = 2a, OS = y, \text{ then } CH = \left(\frac{OC \times FH}{OS} = \right)$$

$$\frac{2ar}{y}, \text{ and } OH = \frac{ry - 2ar}{y} \Big) z = \frac{r^4 + r^2y^2 - 4ar^2y + 4a^2r^2}{r^2 + y^2};$$

therefore $r^4 + r^2y^2 - 4ar^2y + 4a^2r^2 = r^2z + y^2z$. Con-

sequently $2r^2yy - 4ar^2y = 2yzy + y^2z + r^2z$. But z is a mi-

nimum; therefore $rry = 2ar^2 + xy$; consequently $(z =)$

$$\frac{r^2y - 2ar^2}{y} = \frac{r^4 + r^2y^2 - 4ar^2y + 4a^2r^2}{rr + yy}.$$

Hence $yy - 2ay = rr$; and making $OQ = QD = a$, then
 $(y - a =) QS = (\sqrt{rr + aa} =) QC.$

To the same Scholium, page 94.

On the right line BC (Pl. 19, Fig 3) suppose the parallelograms $BG\gamma b$, $MN\nu m$, of the least breadth, to be erected, whose heights BG , MN , their distance Mb , and half the sum of their bases $\frac{1}{2}Mm + \frac{1}{2}Bb = a$, are given: let half the difference of the bases $\frac{1}{2}Mm - \frac{1}{2}Bb$ be called x , let G and N be points in the curve GND ; and producing $b\gamma$ and $m\nu$ to g and n (so that $\gamma g = \nu n = b$), the points g and n may also be in the same curve.

Now if the figure $CDNGB$, revolving about the axis BC , generates a solid, and that solid moves forwards in a rare and elastic medium from C towards B (the position of the right line BC remaining the same), then will the sum of the resistances against the surfaces generated by the lineolæ Gg , Nn , be the least possible, when $\overline{Gg}^4$ is to $\overline{Nn}^4$ as $BG \times Bb$ to $MN \times Mm$.

For the force of a particle on Gg and Nn to move them in the direction BC is as $\frac{1}{\overline{Gg}^2}$ and $\frac{1}{\overline{Nn}^2}$, and the number of particles that strike in the same time on the surfaces generated by Gg and Nn are as (the annuli described by $g\gamma$ and $n\nu$, that is, as $BG \times g\gamma$ and $MN \times n\nu$, or as) BG and MN ; therefore the resistances against those surfaces are as $\frac{BG}{\overline{Gg}^2}$ to $\frac{MN}{\overline{Nn}^2}$, that is (putting y for $\overline{Gg}^2$, and z for $\overline{Nn}^2$), as $\frac{BG}{y}$ to $\frac{MN}{z}$.

But the sum of these resistances $(\frac{BG}{y} + \frac{MN}{z})$ is a minimum. Therefore $-BG \times \frac{\dot{y}}{yy} - MN \times \frac{\dot{z}}{zz} = 0$, or $MN \times \frac{\dot{z}}{zz} = -BG \times \frac{\dot{y}}{yy}$. But $y = (\overline{Gg}^2 = \overline{Bb}^2 + \gamma g^2 =) aa$

$-2ax + xx + bb$; and $z = (\overline{Nn^2} = \overline{Mm^2} + \overline{vn^2}) = aa + 2ax + xx + bb$; therefore $\dot{y} = 2xx - 2ax$, and $\dot{z} = 2ax + 2x\dot{x}$; consequently $\frac{MN}{zz} + 2\dot{x} \times \overline{a+x} = \frac{BG}{yy} \times 2\dot{x} \times \overline{a-x}$; or $\left(\frac{MN}{zz} \times \overline{a+x} = \right) \frac{MN}{zz} \times Mm = \frac{BG}{yy} \times \overline{a-x} = \frac{BG}{yy} \times Bb$. Therefore $(yy) \overline{Gg^4} : (zz) \overline{Nn^4} :: BG \times Bb : MN \times Mm$.

Consequently, that the sum of the resistances against the surfaces generated by the lineolæ Gg and Nn may be the least possible, $\overline{Gg^4}$ must be to $\overline{Nn^4}$ as GBb to NMm.

Wherefore, if γg be made equal to γG , so that the angle γGg may be 45° , and the angle BGg 135° ; also $\overline{Gg^2} = \overline{2\gamma g^2}$, and $\overline{Gg^4} = \overline{4\gamma g^4}$; then $\overline{4\gamma g^4} : \overline{Nn^4} :: GBb : NMm$; and since GR is parallel to Nn, and BG, BR parallel to $\nu\gamma$, $N\nu$; also $\nu\gamma = g\gamma = \gamma G$; it follows that $(\nu\gamma = \gamma G =) Bb : (N\nu =) Mm :: BG : BR$, therefore $Bb = \frac{BG \times Mm}{BR}$;

also $(\nu\gamma =) \gamma G : Nn :: BG : GR$. Consequently $\frac{\overline{4\gamma g^4}}{\overline{Nn^4}} =$

$\frac{4\overline{BG^4}}{\overline{GR^4}} = \left(\frac{GBb}{NMm} = \right) \frac{\overline{BG^2}}{MN \times BR}$. Therefore $4\overline{BG^2} \times BR$ is

to $\overline{GR^3}$ as GR to MN.

END OF VOL. II.

INDEX TO VOL. II.

	Page
ÆQUINOXES , their præcession	252
—— the cause of that motion shewn.....	197
—— the quantity of that motion computed from the causes	252
AIR , its density at any height collected by proposition 22, book 2, and its density at the height of one semidiameter of the earth, shewn	290
—— its elastic force, what cause it may be attributed to..	60
—— its gravity compared with that of water.....	290
—— its resistance, collected by experiments of pendulums	77
—— the same more accurately, by experiments of falling bodies, and a theory.....	123
ATTRACTION of all bodies demonstrated.....	176
—— the certainty of this demonstration shewn.....	160
—— the cause or manner thereof nowhere defined by the author.....	313
—— the common centre of gravity of the earth, sun, and all the planets, is at rest, confirmed by corollary 2, pro- position 14, book 3.....	182
—— the common centre of gravity of the earth and moon goes round the <i>orbis magnus</i>	183
—— its distance from the earth and from the moon....	245
COMETS , a sort of planets, not meteors.....	261, 285
—— higher than the moon, and in the planetary regions..	254
—— their distance, how collected very nearly by obser- vations	255
—— more of them observed in the hemisphere towards the sun than in the opposite hemisphere; and how this comes to pass.....	260
—— shine by the sun's light reflected from them.....	<i>ibid.</i>
—— surrounded with vast atmospheres	258, 261
—— those which come nearest to the sun probably the least	297

INDEX.

	Page
COMETS, why they are not comprehended within a zodiac, like the planets, but move differently into all parts of the heavens	307
—— may sometimes fall into the sun, and afford a new supply of fire.....	<i>ibid.</i>
—— the use of them hinted.....	294
—— move in conic sections, having their foci in the sun's centre, and by radii drawn to the sun describe areas proportional to the times. Move in ellipses if they come round again in their orbits, but these ellipses will be near to parabolas.....	262
COMET's parabolic trajectory found from three observations given	268
—— corrected when found.....	298
—— place in a parabola found to a given time.....	262
—— velocity compared with the velocity of the planets..	<i>ibid.</i>
—— TAILS directed from the sun.....	289
—— brightest and largest immediately after their passage through the neighbourhood of the sun.....	286
—— their wonderful rarity.....	291
—— their origin and nature.....	258
—— in what space of time they ascend from the heads	291
COMET of the years 1664 and 1665	
—— the observations of its motion compared with the theory.....	299
—— of the years 1680 and 1681	
—— observations of its motion	271
—— its motion computed in a parabolic orbit.....	276
—— in an elliptic orbit.....	278
—— its trajectory, and its tail in the several parts of its orbit, delineated	283
—— of the year 1682	
—— its motion compared with the theory.....	305
—— seems to have appeared in the year 1607, and likely to return again after a period of 75 years.....	306, 307
—— of the year 1683	
—— its motion compared with the theory.....	304

INDEX.

	Page
COMET of the year 1723	
—— its motion compared with the theory.....	306
CURVATURE of figures how estimated.....	25, 209
DESCENT of heavy bodies in vacuo, how much it is.....	187
—— and ascent of bodies in resisting mediums	2, 17, 18, 37, 39, 115
EARTH, its dimension by <i>Norwood</i> , by <i>Picart</i> , and by <i>Cassini</i>	187
—— its figure discovered, with the proportion of its diameters, and the measure of the degrees upon the meridian.	187, 191
—— the excess of its height at the equator above its height at the poles.....	190, 196
—— its greatest and least semidiameter.....	190
—— its mean semidiameter.....	<i>ibid.</i>
—— the globe of, more dense than if it was entirely water	180
—— the nutation of its axis.....	197
—— the annual motion thereof in the <i>orbis magnus</i> demonstrated.....	301
—— the eccentricity thereof how much.....	245
—— the motion of its aphelion how much.....	186
FLUIDS, the laws of their density and compression shewn, <i>Sect. V</i>	50
—— their motion in running out at a hole in a vessel determined	98
FORCE, centrifugal, of bodies on the earth's equator, how great.....	187
GOD, his nature.....	312
GRAVITY, of a different nature from magnetical force.....	176
—— the cause of it not assigned.....	313
—— tends towards all the planets.....	171
—— from the surfaces of the planets upwards decreases in the duplicate ratio of the distances from the centre.....	180
—— from the same downwards decreases nearly in the simple ratio of the same	<i>ibid.</i>
—— tends towards all bodies, and is proportional to the quantity of matter in each.....	176
—— is the force by which the moon is retained in its orbit	169
—— proved by an accurate calculus.....	245, 246

INDEX.

	Page
GRAVITY, is the force by which the primary planets and the satellites of Jupiter and Saturn are retained in their orbits	171
HEAT, an iron rod increases in length by	196
—— of the sun, how great at different distances from it..	285
—— how great in Mercury	180
—— how great in the comet of 1680, when in its perihelion	285
HEAVENS are void of any sensible resistance, 181, 261, 293 ; and, therefore, of almost any corporeal fluid what-ever.....	128, 129
—— suffer light to pass through them without any refraction	286
HYDROSTATICS, the principles thereof delivered.. <i>Sect. V.</i>	50
HYPOTHESES of what kind soever rejected from this philosophy.....	314
JUPITER, its periodic time.....	165
—— its distance from the sun.....	166
—— its apparent diameter.....	163
—— its true diameter.....	179
—— its attractive force, how great.....	178
—— the weights of bodies on its surface.....	179
—— its density and quantity of matter	<i>ibid.</i>
—— its perturbation by Saturn, how much.....	183
—— the proportion of its diameters exhibited by computation, and compared with observations.....	191
—— its rotation about its axis, in what time performed..	<i>ibid.</i>
—— the cause of its belts hinted at.....	261
LIGHT, not caused by the agitation of any ethereal medium.	144
MAGNETIC force.....	62, 176, 247
MARS, its periodic time.....	165
—— its distance from the sun.....	166
—— the motion of its aphelion.....	186
MATTER, its extension, hardness, impenetrability, mobility, <i>vis inertiae</i> , gravity, how discovered.....	161
—— subtle, of <i>Descartes</i> , enquired into	84
MERCURY, its periodic time.....	165
—— distance of from the sun.....	166
—— the motion of its aphelion.....	186
METHOD of fluxions	13
—— differential	263

INDEX.

	Page
METHOD of finding the quadratures of all curves very nearly true	264
—— of converging series applied to the solution of difficult problems.....	25, 225
MOON, the figure of its body collected by calculation.....	247
—— its librations explained.....	186
—— its mean apparent diameter and true diameter	245
—— weight of bodies on its surface.....	<i>ibid.</i>
—— its density and quantity of matter	<i>ibid.</i>
—— its mean distance from the earth, how many greatest semidiameters of the earth contained therein	<i>ibid.</i>
—— how many mean semidiameters.....	247
—— its force to move the sea how great	241
—— not perceptible in experiments of pendulums, or any statal or hydrostatal observations	244
—— its periodic time	246
—— the time of its synodical revolution.....	208
—— its motions, and the inequalities of the same derived from their causes.....	197, 234
—— revolves more slowly, in a dilated orbit, when the earth is in its perihelion; and more swiftly in the aphelion the same, its orbit being contracted	197, 234, 235
—— revolves more slowly, in a dilated orbit, when the apogæon is in the syzygies with the sun; and more swiftly, in a contracted orbit, when the apogæon is in the quadratures.....	236
—— revolves more slowly, in a dilated orbit, when the node is in the syzygies with the sun; and more swiftly, in a contracted orbit, when the node is in the quadratures..	237
—— moves slower in its quadratures with the sun, swifter in the syzygies; and by a radius drawn to the earth describes an area, in the first case less in proportion to the time, in the last case greater	197
—— the inequality of those areas computed.....	206
—— its orbit is more curve, and goes farther from the earth in the first case; in the last case its orbit is less curve, and comes nearer to the earth	197
—— the figure of this orbit, and the proportion of its diameters collected by computation	209

INDEX.

Page

MOON , a method of finding the distance of, from the earth	
by its horary motion	209
—— its apogæon moves more slowly when the earth is in its aphelion, more swiftly in the perihelion	198, 235
—— its apogæon goes forward most swiftly when in the syzygies with the sun, and goes backward in the quadratures	198, 237
—— its eccentricity greatest when the apogæon is in the syzygies with the sun; least when the same is in the quadratures	198, 237
—— its nodes move more slowly when the earth is in its aphelion, and more swiftly in the perihelion	198, 235
—— its nodes are at rest in their syzygies with the sun, and go back most swiftly in the quadratures	198
—— the motions of the nodes and the inequalities of its motions computed from the theory of gravity .214, 218, 222, 225	
—— the same from a different principle	226
—— the variations of the inclination computed from the theory of gravity	230, 232
—— the equations of its motions for astronomical uses...	235
—— the annual equation of its mean motion	<i>ibid.</i>
—— the first semi-annual equation of the same	236
—— the second semi-annual equation of the same	237
—— the first equation of its centre	238
—— the second equation of its centre	239
—— its first variation	212
—— the annual equation of the mean motion of its apogee	235
—— the semi-annual equation of the same	237
—— the semi-annual equation of its eccentricity	<i>ibid.</i>
—— the annual equation of the mean motion of its nodes .	235
—— the semi-annual equation of the same	226
—— the semi-annual equation of the inclination of the orbit of, to the ecliptic	234
—— the method of fixing the theory of the lunar motions from observations	259
MOTION of bodies resisted in the ratio of the velocities, <i> Sect. I.</i>	1

INDEX.

	Page
MOTION of bodies resisted in the duplicate ratio of the velocity.....	Sect. II. 9
—— partly in the simple and partly in the duplicate ratio of the same.....	Sect. III. 34
—— of bodies proceeding by their <i>vis insita</i> alone in resisting mediums	1, 2, 9, 11, 34, 35, 96
—— of bodies ascending or descending in right lines in resisting mediums, and acted on by an uniform force of gravity	2, 17, 18, 37, 39
—— of bodies projected in resisting mediums, and acted on by an uniform force of gravity.....	5, 21
—— of bodies revolving in resisting mediums...Sect. IV.	43
—— of funependulous bodies in resisting mediums, Sect. VI.	62
—— and resistance of fluids.....Sect. VII.	87
—— propagated through fluids.....Sect. VIII.	130
—— of fluids after the manner of a vortex, or circular, Sect. IX.	146
PENDULUMS, their properties explained	Sect. VI. 62
——, the diverse lengths of isochronous, in different latitudes, compared among themselves, both by observations and by the theory of gravity.....	192 to 197
PLANETS not carried about by corporeal vortices	157
PLANETS, <i>Primary</i> , surround the sun.....	165
—— move in ellipses whose focus is in the sun's centre ..	183
—— by radii drawn to the sun describe areas proportional to the times	166, 183
—— revolve in periodic times that are in the sesquiplicate proportion of the distances from the sun	165
—— are retained in their orbits by a force of gravity which respects the sun, and is reciprocally as the square of the distance from the sun's centre	167, 171
PLANETS, <i>Secondary</i> , move in ellipses having their focus in the centre of the primary.....	197
—— by radii drawn to their primary describe areas proportional to the times	162, 164, 167
—— revolve in periodic times that are in the sesquiplicate proportion of their distances from the primary	162, 164

INDEX.

	Page
PLANETS, their periodic times	165
—— their distances from the sun	166
—— the aphelia and nodes of their orbits do almost rest.	185
—— their orbits determined	186
—— the way of finding their places in their orbits. 117 to	122
—— their density suited to the heat they receive from the sun	180
—— their diurnal revolutions equable	186
—— their axes less than the diameters that stand upon them at right angles	187
PROJECTILES move in parabolas when the resistance of the medium is taken away	27
—— their motions in resisting mediums	5, 21
PULSES of the air, by which sounds are propagated, their intervals or breadths determined.....	144, 146
—— these intervals in sounds made by open pipes pro- bably equal to twice the length of them	<i>ibid.</i>
QUALITIES of bodies how discovered, and when to be sup- posed universal	160
RESISTANCE, the quantity thereof in mediums not continued	95
—— in continued mediums	192
—— in mediums of any kind whatever.....	97
—— of mediums is as their density, <i>cæteris paribus</i> 84, 85, 89, 95, 113, 128	
—— is in the duplicate proportion of the velocity of the bodies resisted, <i>cæteris paribus</i>	9, 75, 89, 95, 113, 122
—— is in the duplicate proportion of the diameters of spherical bodies resisted, <i>cæteris paribus</i>	79, 81, 95, 113
—— of fluids threefold, arises either from the inactivity of the fluid matter, or the tenacity of its parts, or friction ...	42
—— found in fluids, almost all of the first kind	85, 127
—— cannot be diminished by the subtilty of the parts of the fluid, if the density remain	128
—— of a globe in mediums not continued	95
—— in compressed mediums	113

INDEX.

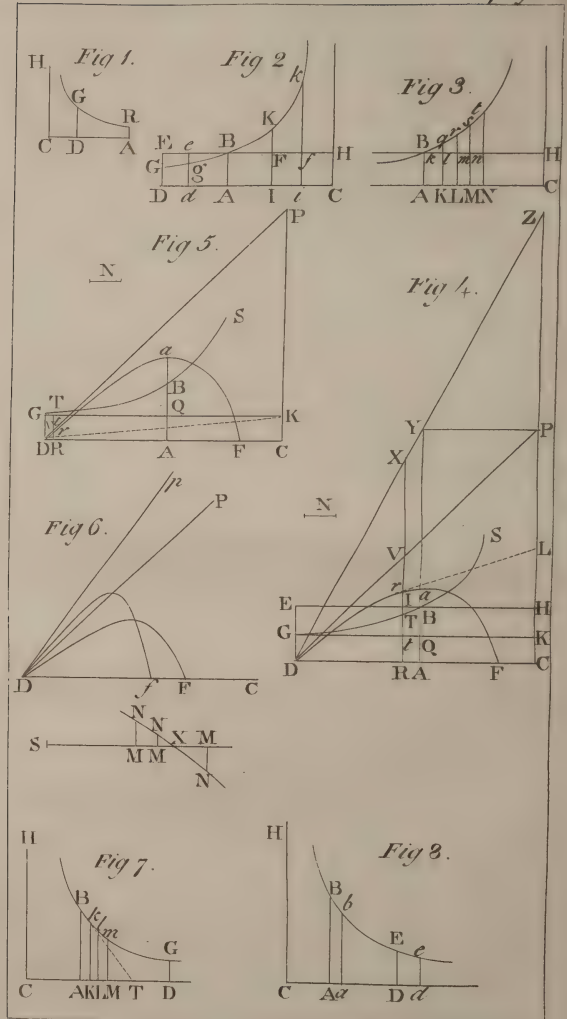
	Page
RESISTANCE of a globe, what proportion it bears to that of a cylinder, in mediums not continued	92
—— in compressed mediums	112
—— how found by experiments	115 to 128
—— to a frustum of a cone, how made the least possible.	94
—— what kind of solid it is that meets with the least....	95
RESISTANCES, the theory thereof confirmed by experiments of pendulums	74 to 85
—— by experiments of falling bodies	115 to 129
RULES of philosophy	160
SATELLITES, the greatest heliocentric elongation of Jupiter's	163
—— the greatest heliocentric elongation of the <i>Huygenian</i> satellite from Saturn's centre	178
—— the periodic times of Jupiter's, and their distances from his centre	162, 163
—— the periodic times of Saturn's, and their distances from his centre	164, 165
—— the inequalities of the motions of Jupiter and Saturn derived from the motions of the moon	197
SATURN, its periodic time.....	165
—— its distance from the sun	166
—— its apparent diameter.....	165
—— its true diameter	179
—— its attractive force how great	178
—— the weight of bodies on its surface	179
—— its density and quantity of matter.....	<i>ibid.</i>
—— its perturbation by the approach of Jupiter how great	184
—— the apparent diameter of its ring	165
SHADOW of the earth to be augmented in lunar eclipses, because of the refraction of the atmosphere	239
SOUNDS, their nature explained, 133, 138, 140, 142, 143, 144, 145 —— not propagated in <i>directum</i>	132
—— caused by the agitation of the air.....	144
—— their velocity computed	144, 145
—— somewhat swifter by the theory in summer than in winter	146

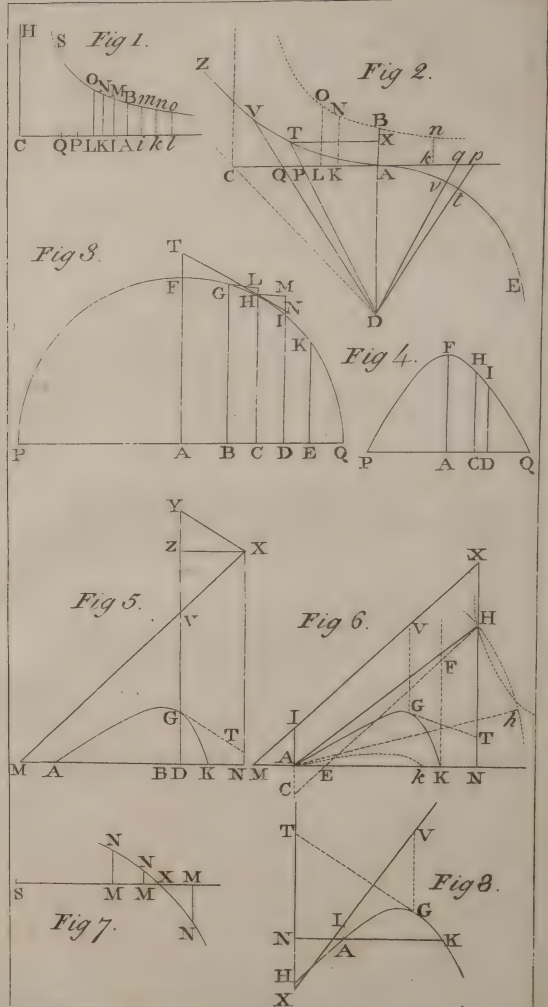
INDEX.

	Page
SOUNDS, cease immediately, when the motion of the sonorous body ceases	140
— how augmented in speaking trumpets	146
SPACE, not equally full	175
SPIRAL cutting all its radii in a given angle, by what law of centripetal force, tending to the centre thereof, it may be described by a revolving body	43, 48
SPIRIT pervading all bodies, and concealed within them, hinted at, as required to solve a great many phænomena of Nature	314
STARS, the fixed, demonstrated to be at rest	185
— their twinkling what to be ascribed to	287
— new, whence they may arise	307
SUBSTANCES of all things unknown	313
SUN, moves round the common centre of gravity of all the planets	182
— the periodic time of its revolution about its axis	186
— its mean apparent diameter	245
— its true diameter and horizontal parallax	178
— has a menstrual parallax	184
— its attractive force how great	178
— the weight of bodies on its surface	179
— its density and quantity of matter	<i>ibid.</i>
— its force to disturb the motions of the moon	169, 205
— its force to move the sea	240
TIDES of the sea derived from their cause	199, 240, 241
VACUUM, a, proved, or that all spaces, if said to be full, are not equally full	175
VELOCITY, the greatest that a globe falling in a resisting medium can acquire	113
VENUS, its periodic time	165
— its distance from the sun	166
— the motion of its aphelion	186
VORTICES, their nature and constitution examined	309
WAVES, the velocity with which they are propagated on the superficies of stagnant water	136

INDEX.

	Page
WEIGHTS of bodies towards the sun, the earth, or any planet, are, at equal distances from the centre, as the quantities of matter in the bodies	172
—— do not depend upon the forms and textures of bodies	173
—— of bodies in different regions of the earth found out, and compared together	192







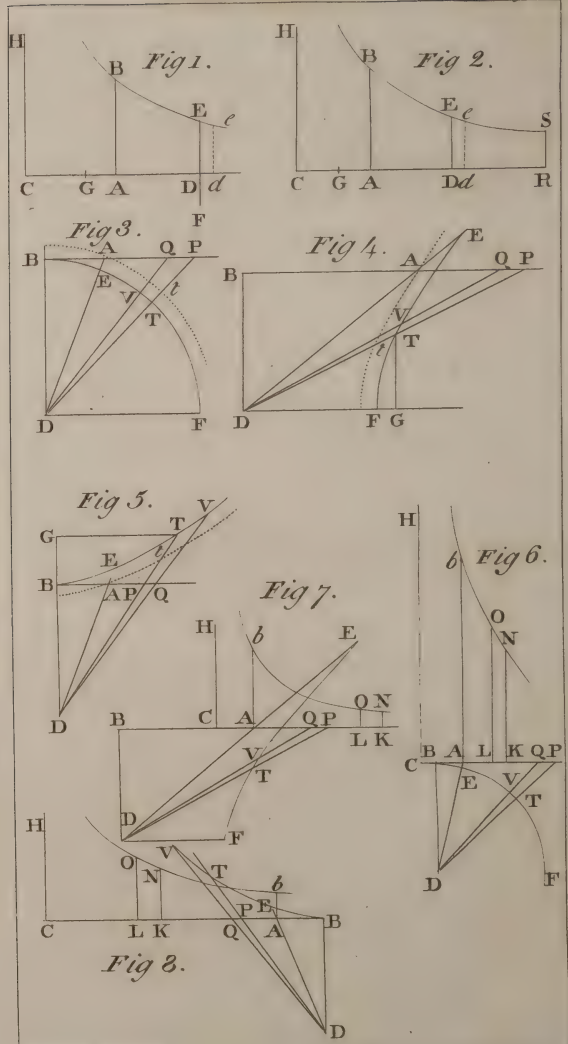


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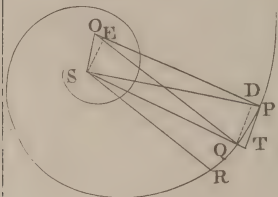


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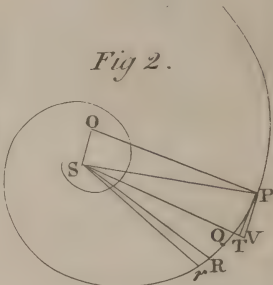


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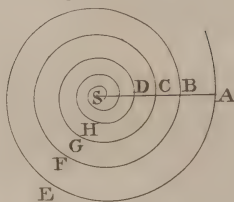


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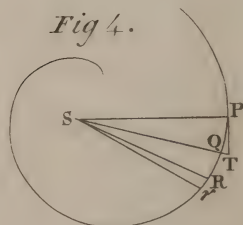


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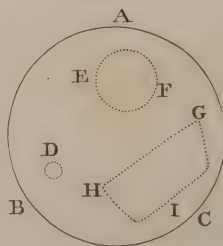
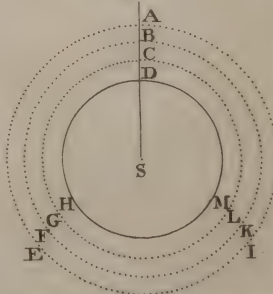


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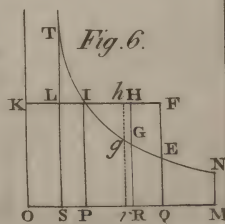
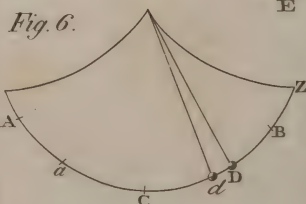
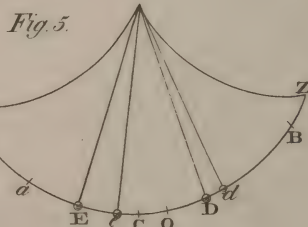
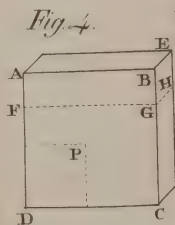
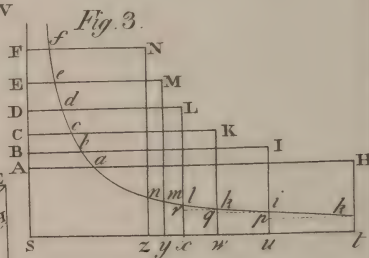
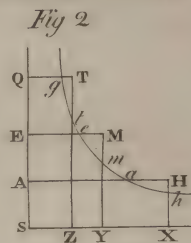
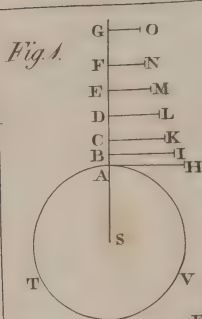


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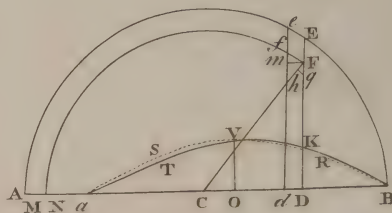


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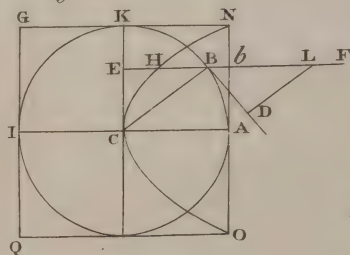


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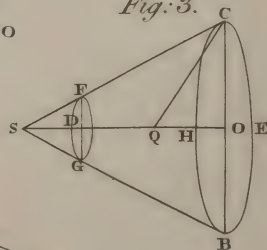
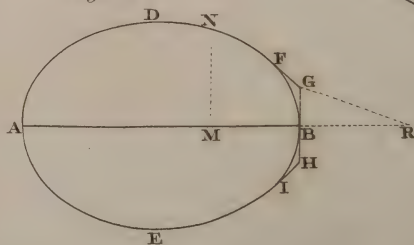
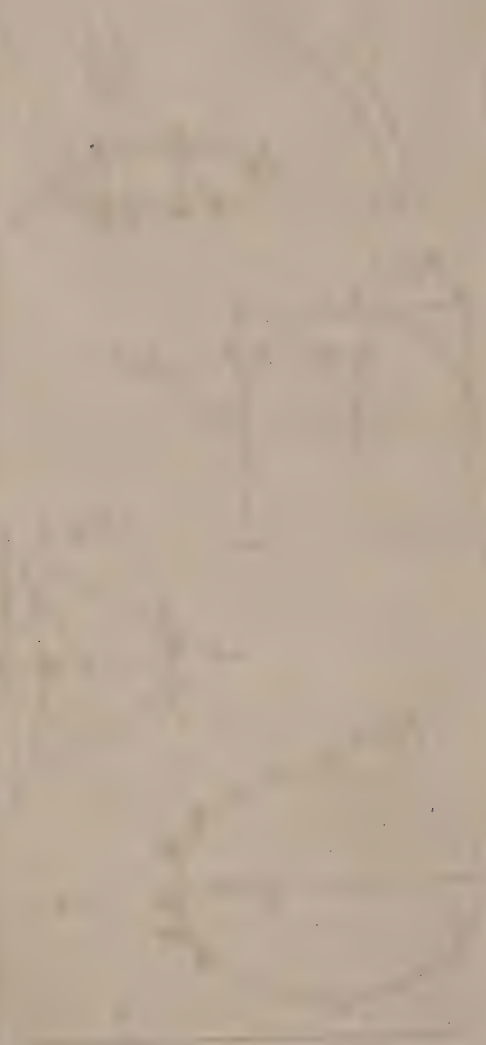


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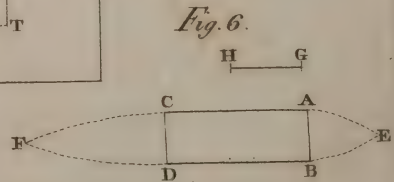
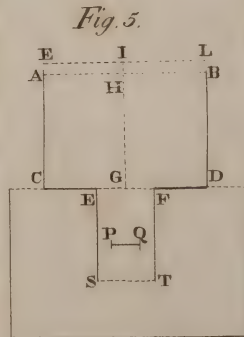
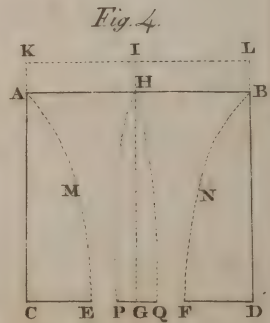
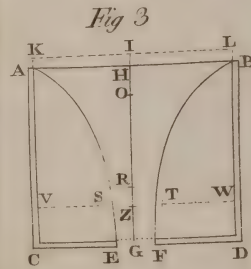
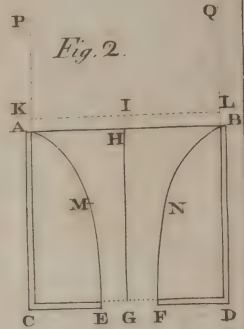
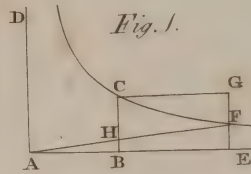


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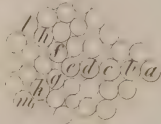


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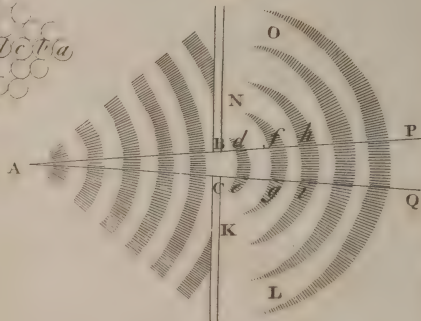


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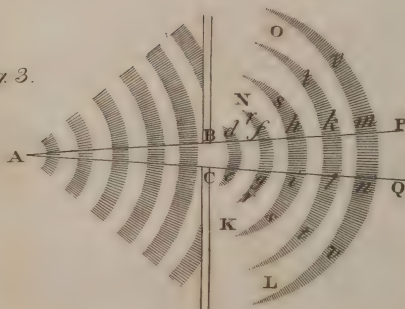


Fig. 4.

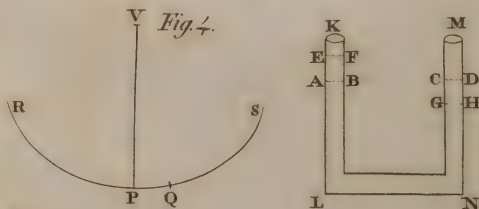
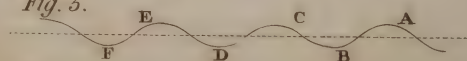
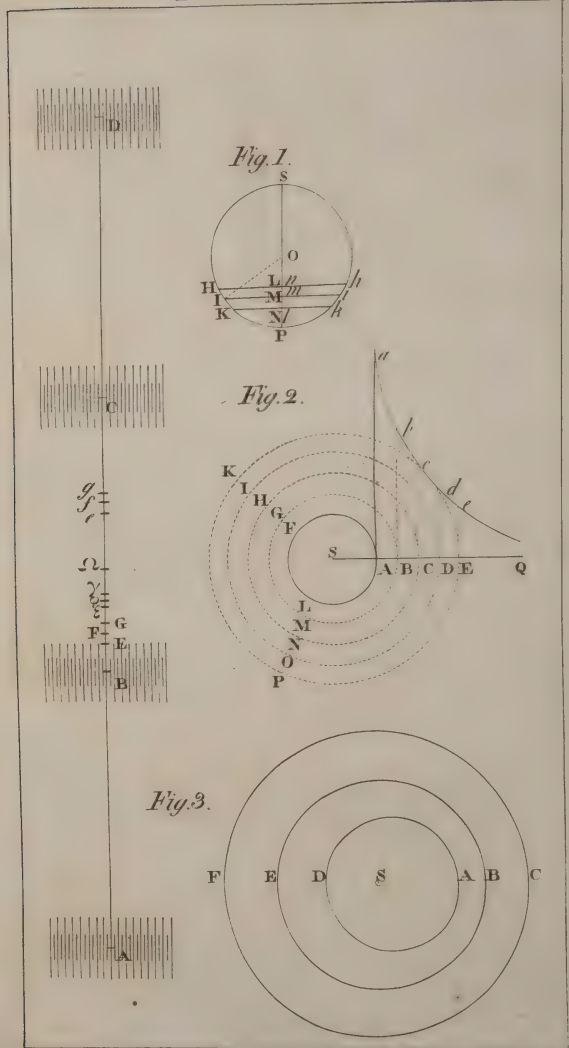


Fig. 5.



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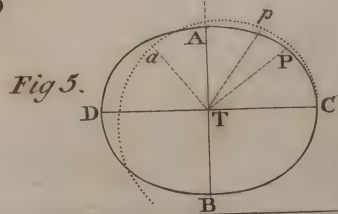
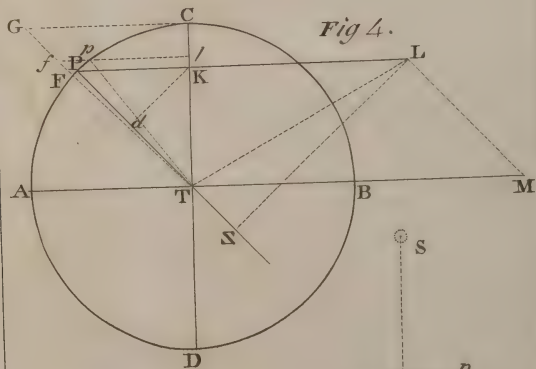
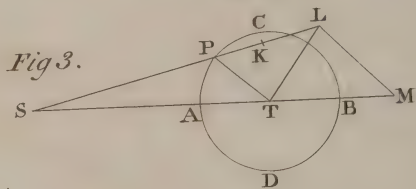
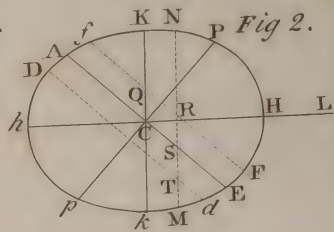
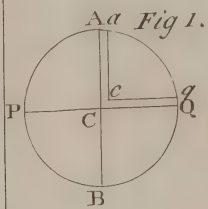
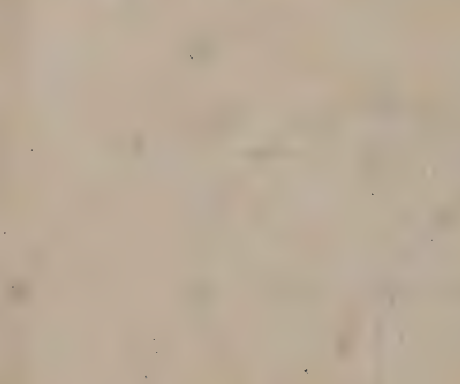
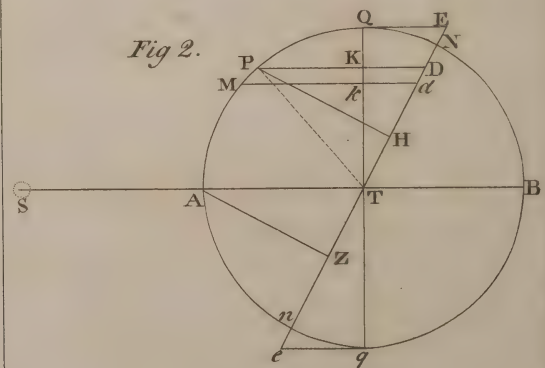
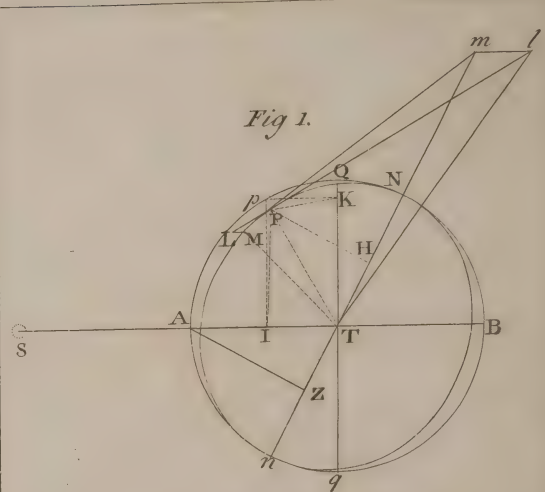


Figure 1

Figure 2





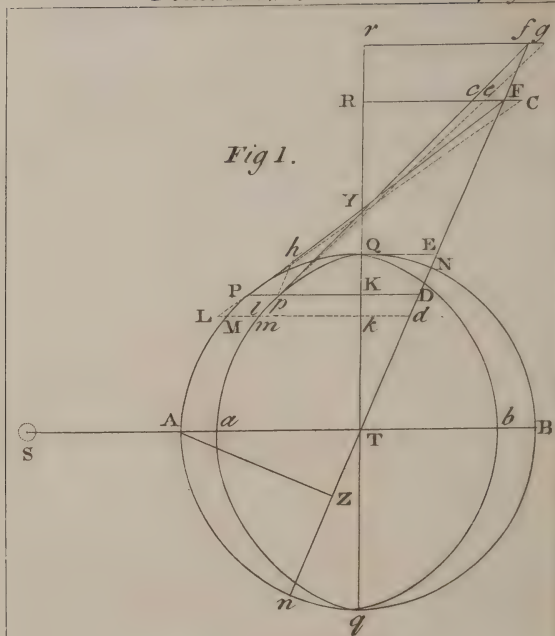


Fig 1.

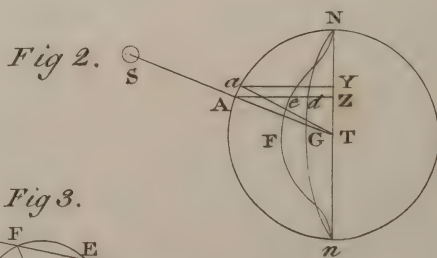


Fig 2.

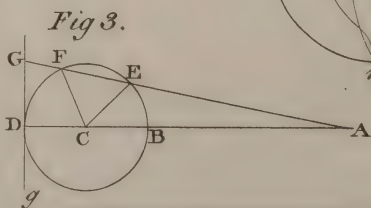


Fig 3.

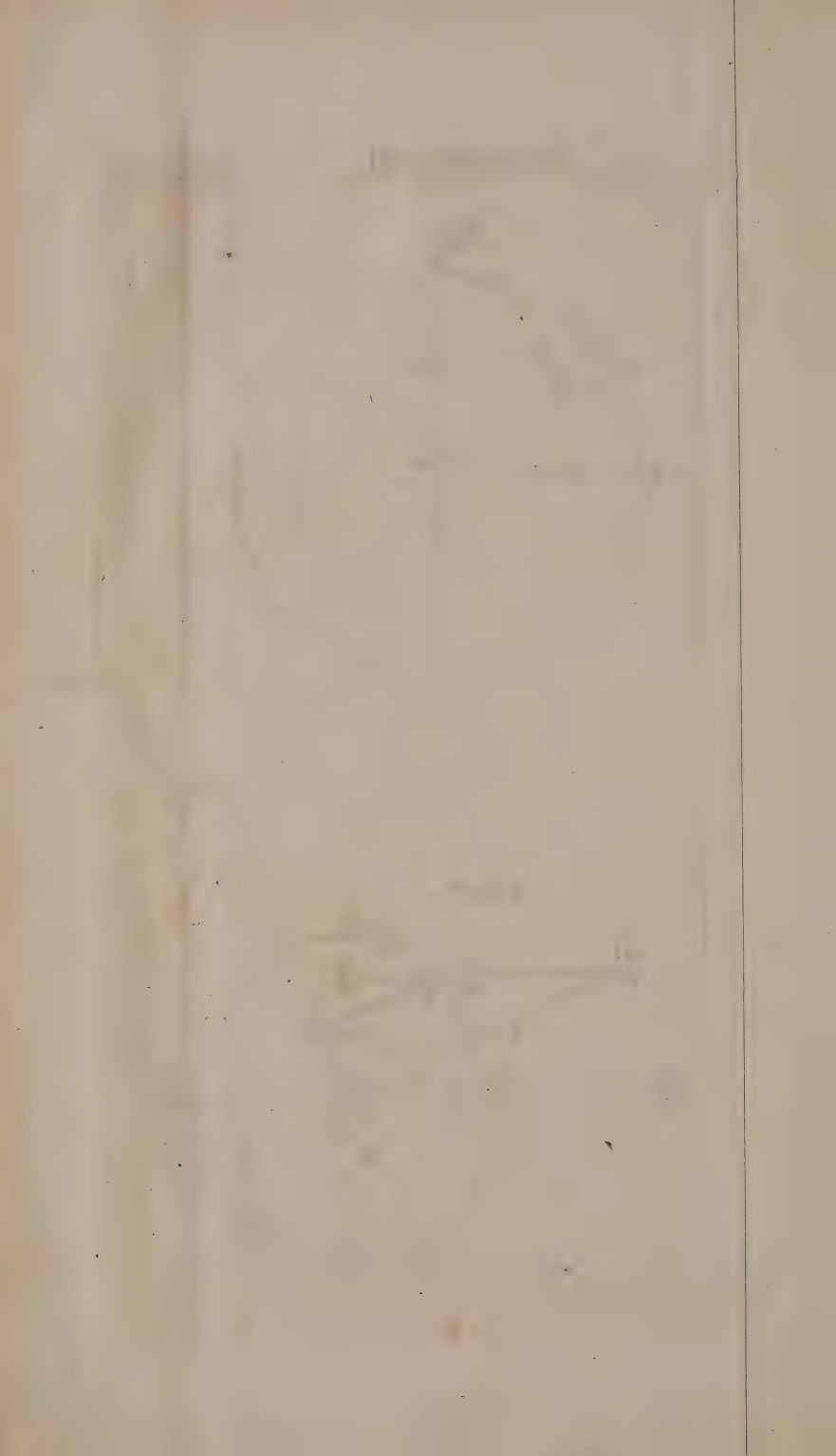


Fig 1.

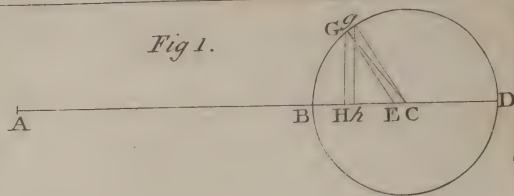


Fig 2.

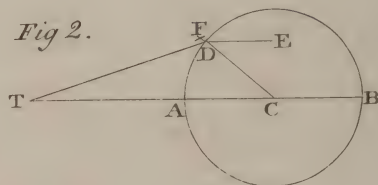


Fig 3.

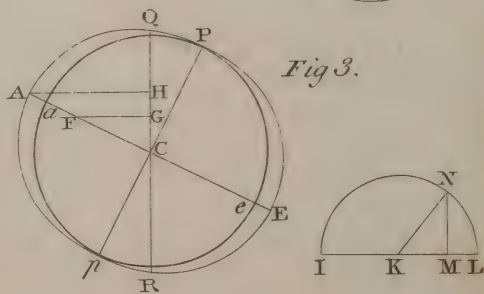


Fig 4.

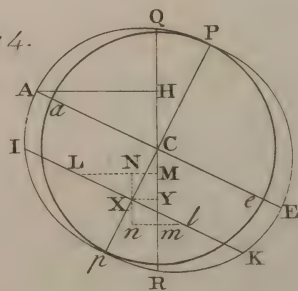


Fig. 1.

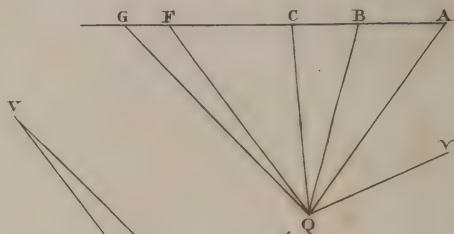


Fig. 2.

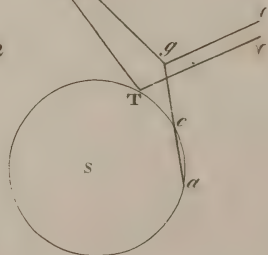


Fig. 3.

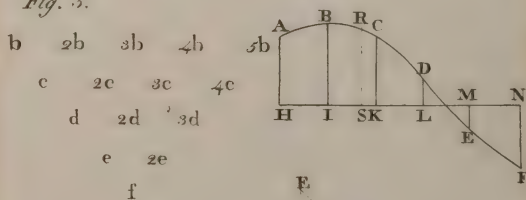
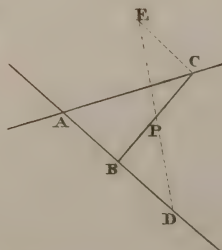


Fig. 4.



2-1-21



